CHAPTER V

Mathematics in Secondary Schools

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1. Mathematics teachers are at present doing what philosophers urge -- asking what we do, why and whether we should. In an age of calculating machines need children know multiplication tables? Should we teach manipulation in algebra? If so, how much? Are logarithms obsolete? Should we teach trigonometry at all? If so, why? What parts of recent mathematics ought to come into the school syllabus? What should go out to make room for new topics? More fundamental, what considerations should determine our choice of syllabus?

2. Such questions deal with *what* is taught. Perhaps even more important is *how* we teach. Should mathematics fuse with science and the study of the environment, or should it be taught as abstractly as possible? Should it be formal or informal, rigorous or intuitive? In what proportions should discovery and "telling" be mixed?

3. A third type of question deals with how we achieve our objectives. It is easy to deliver a very inspiring sermon on the ideal school without explaining how to get there. In order to avoid a sense of unreality I will begin with some notes on the mechanics of educational advance.

4. I suppose that if you sent a watch to be repaired and it came back with the mainspring broken, whatever improvements the watchmaker might have made in the gear train you would feel he had done a bad job. But too often something of that kind happens in mathematical education. Most adults after leaving school not only know very little mathematics; they are incapable of thinking about it and afraid to try. Miss Biggs will surely stress in her talk that young children do not start out like that; they are interested in mathematics and willing to think about it. Accordingly, even for a child of rather low intelligence, the main difficulty in mathematics lies neither in the nature of the subject nor in the limitations of the learner, but rather in the attitude of the adults around him.

5. It is often thought that the solution lies with teacher training colleges. But this is somewhat naive. If a student comes to a college feeling critical of his own schooling, the college may be able to help him. But many students by 18 years of age have already a firm picture of what education is; it is what they had as pupils. They will return to school and teach as they were taught.

6. One of the most obvious (and least recognized) facts is the interlocking of the entire educational system. When universities criticize secondary schools they seem

only partly aware of the immense responsibility of universities for the state of secondary education. When secondary school teachers criticize elementary schools they do not always stress that elementary teachers are the products of secondary schools.

7. Mathematics is an activity and depends greatly on the attitude of the learner. Attitudes are formed young. The greatest influence on the future teacher is the elementary school; the next greatest, the secondary school.

8. One way in which elementary education concerns secondary teachers is that, in most countries, there is a movement of mathematical topics downwards. Secondary teachers complain they have not time to handle all the topics the university is handling over to them. The main reason is often a mathematical vacuum in the elementary schools. A.P. Rollett, in an article on English schools, mentioned abler children who did not start an intensive attack on algebra and geometry until they were 11 years old; he said they endured "mathematical stagnation". In North America, where a much milder exposure to mathematics used to begin at 14 years of age, these children would have been described as fiercely "accelerated". It is extremely desirable that a much richer content should be available to children 9-13 years of age, not merely to ease the secondary syllabus, but because these are the years in which the stimulus of new ideas most readily creates lasting intellectual interests. Secondary teachers should therefore be preparing for a steady transfer of serious algebra and geometry to elementary schools.

9. Secondary teachers will object that elementary teachers cannot handle such topics properly. But this ignores the fact that the future elementary teachers are to-day sitting in secondary classrooms. I would suggest as a *minimum programme* for secondary mathematics that every pupil should leave school with the basic equipment of a good elementary school mathematics teacher.

10. This means the following objectives, in the following order of importance. (1) The pupil should enjoy mathematics and not be afraid of thinking about it. (2) He should be able to grasp mathematical results informally, pictorially or in terms of a concrete situation. (3) He should have worked with mathematics in connection with simple scientific laws and with the environment generally. (4) In this way he should have acquired as much knowledge and understanding of arithmetic, algebra, geometry (and perhaps other mathematical subjects) as possible.

11. The application of this idea will naturally vary from country to country and from place to place. If elementary schools are already producing enthusiastic and active mathematicians, the secondary school should make sure it does not spoil what has been achieved, and should find it easy to build further. If children enter secondary school already disliking mathematics, a very radical rehabilitation procedure may be called for. If the secondary school was able to improve their attitude and give them a confident grasp of the topics mentioned in *Mathematics in Primary Schools** it could feel well pleased with itself.

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12. Some secondary teachers have a very formal approach to mathematics and would be incapable of carrying out such a programme. The work should be done wherever there are teachers capable of it, and the authorities should emphasize that this is work of first importance, and the key to the whole national advance in mathematical education.

13. Mathematics involves general intelligence. The accomplishments of individual children will naturally vary. Complete success would mean that a child showed the same level of interest, confidence, initiative, originality and ingenuity in tackling mathematical and scientific problems as he did in any other department of work or play. Success would be judged not by multiple choice tests but by observing the child in a real situation.

14. Children, of course, would have to work at their own pace. I once ran a mathematics club in New Zealand where children could pass little tests rather like those for scout badges. Many of these were for *basic understanding of an idea,* e.g. "understands the use of x". In some American schools, each child has a folder which records his advance in reading. The same could be done with mathematics. This idea is elaborated somewhat in the Ontario report on Geometry, K-13.

15. This brings me to an administrative point. When I came to North America I saw something I had never seen before, a textbook labelled "Grade 9 Mathematics" – a scheme of work to be covered by all children in a certain type of class (e.g. academic, technical); a scheme laid down by the education authority and often enforced by inspectors. A teacher was thus under pressure to "cover the course", regardless of the fact that the slower pupils were bewildered and the quicker ones bored. Good teaching of mathematics in such an administrative setting is next to impossible. A real teacher is forced to become an underground worker.

16. Uniformity is particularly out of place when attempts are being made to change or to enlarge the syllabus. Inevitably, there will only be a limited number of schools where teachers are able to teach the new material effectively and inspiringly. These should not be held back until the whole city, province or country is ready – which it never will be. Rather, the strongest centres should go ahead and the influence allowed to spread gradually to others.

17. In a transitional period all kinds of informal measures may be necessary. A secondary school might (as has been tried in some places) offer a mathematics club, say once every 2 weeks, to the ablest and most interested pupils in the elementary schools near it. Secondary school pupils may meet two or three elementary school children and help them with algebra and geometry; in this way teaching talent may become recognized and an excellent training given to the future teacher. The ablest children in elementary schools should be encouraged to read ahead on their own. A survey should be made of what books children in fact can read successfully. On the analogy of music teachers, a good mathematics teacher might tour a number of elementary schools, giving one lesson a week in each to promote interest in the subject.

18. I have spent some time here discussing ages 9-13 because I am more and more convinced that this is the vital strategic area in which good teachers should be concentrated. The ages 0-8 are of course very important. However I am here discussing the role of the very best secondary teachers, those with a good knowledge of mathematics and the ability to bring out its interest and simplicity. I do not think they can do much by spending time with children below 9 years of age. The beginnings of arithmetic move very slowly. Also the work is simple; elementary school teachers can cope with it, once they have been persuaded that arithmetic is something you can experiment with and think about rather than learn purely by rote. On the other hand, if children from 9 to 13 are fully extended, many of them will get deep into fairly technical mathematics. Much of it they may read and do for themselves, but from time to time they will need advice which, in much of the world to-day, elementary teachers are not able to give. Occasional contact with the best secondary teachers will enable at any rate the strongest pupils to forge ahead. In the future some of them will become teachers and they will then regard it as normal to cover a substantial part of the present secondary curriculum (of many countries) in elementary school. In this way a richer syllabus will gradually spread and become established.

19. Incidentally, these ideas are not put forward as pure theory. My book, *Vision in Elementary Mathematics,* was based on experience with children in the age range 9-13. In my travels, I have found abundant evidence of children being able to cope with a far richer diet than anything that was being offered to them. For co-operation between secondary and elementary teachers, Barrie, Ontario, would be a good example of a place where this has been going on for several years.

Manipulation, Logarithms, Trigonometry

20. In an age of calculating devices the question arises at every level from arithmetic to calculus – how far need we teach only basic understanding and how far need we be concerned with slickness in manipulation?

21. The answer certainly lies somewhere between the extreme positions.

22. First, we should certainly get rid of the old idea that the use of any helpful device is cheating. When I see on examination papers, "Slide rules not permitted" I wonder if the next line will read, "All calculations to be in Roman numerals".

23. In general, we should be prepared to use, when appropriate, ready reckoners, slide rules, tables of integrals, graphs and any mechanical devices we can buy or make.

24. Some British examinations used to include very tricky and awkward integrals. I see little point in this. A course on calculus should cover the standard processes and the simpler results of integration. It should point out that certain expressions, such as $\sqrt{(1-x^4)}$ and $e-x^2$ do not have elementary integrals, and then set out the 4 main classes that are tractable – rational functions, rational functions of sin x and cos x, and so forth. The arrangement of tables of integrals should be explained.

25. Now, as to the need for some skill in manipulation. A new concept is often best introduced by some simple example involving a certain amount of calculation. In U.S.A., where even the teachers in secondary schools are very weak at algebraic manipulation, I sometimes tried to introduce calculus or matrices by a piece of algebra. Often I found the teachers' mental energy was completely absorbed in understanding the algebra, and they had none for understanding the new concept. Clearly this is an undesirable situation. The same applies at a lower level. Algebra itself can be interestingly introduced by studying some "coincidence" in arithmetic or through some simple scientific law. The pupils are expected to discover the regularity involved. Clearly this useful method of teaching is excluded if the children's arithmetic is too weak.

26. A calculating machine or a ready reckoner will give the answer to a specific problem but it does not help us to observe a law. It will give us the square of a particular number or find a particular product but it does not, when presented with a collection of figures, exclaim "Why, these are all squares" or "This is the 7 times table". Pupils need to become sufficiently familiar with multiplication tables, algebraic formulae and so forth to recognize situations in which these occur or to which they are relevant. Pattern recognition fortunately is an activity children enjoy and one that helps to fix the individual facts in the memory.

27. Even for problem solving certain material aids can be useful. For instance, to attack a problem in a traditional geometry course, a systematic approach is helped by a diagram showing the main theorems of Euclid. Generally pupils should be encouraged to make, and use, summaries of the information at their disposal. (Even many university students do not do this).

28. An approach that both saves time and creates interest is to use future theorems as exercises. (Theorems generally should be presented as problems, and not thought of as a separate category). For instance, in calculus to evaluate

$\int_0^\infty x^n e^{-x}\,dx$ for $n = 0, 1, 2 \ldots$ is a reasonable exercise. But this integral is still meaningful when n is a fraction and allows us to define $n!$ for fractional n.

29. In trigonometry, deriving the formulas for sin 2A and cos 2A can be presented as a problem in co-ordinate geometry; if (c, s) is the point on the unit circle at angle A, what are the co-ordinates of the point at angle 2A?

30. Conversely, more advanced topics give a good excuse for taking another look at earlier questions and treating them more efficiently. It is really remarkable that simple formulas should exist for the sine and cosine of a sum; for most functions, f(A+B) is not at all simply related to f(A) and f(B). Matrices give one way of seeing why such formulas exist and deriving them. With calculus one can make plausible the connection of sine and cosine with e^{ix} and thus reduce all trigonometric identities to algebra.

31. There is the utmost diversity of views on the status of trigonometry. Some topologists, who never use it, regard it as a waste of time. At the other extreme,

most practical people still have to deal with solid objects that have definite shapes and sizes. In its traditional form trigonometry is still relevant for an architect or an astronomer. The trigonometrical functions are relevant to mechanical and electrical oscillations, to complex variable and to Fourier series.

32. I believe there is general agreement that work on the solution of triangles can be pruned drastically, particularly numerical work with logarithms and the related formulae. As a method of calculation, logarithms are nearly obsolete. (They may still be useful for finding a^n with large n). The logarithmic scale is still of significance and is readily introduced in connection with the slide rule. Incidentally the 17th century approach to logarithms was much simpler than the modern one, as logarithms preceded fractional indices by about half a century. There are considerable teaching advantages in defining logarithms first, and then defining x^k, for any k, as the number that occurs k times as far along the slide rule as x. Children are willing to concede that such a number exists, while they may have serious doubts whether $10^{0 \cdot 301}$ means anything at all.

33. I am inclined to regard algebra, co-ordinate geometry of 2 and 3 dimensions, and trigonometry as an indissoluble whole. Trigonometry enters naturally as the means whereby "a distance r at an angle θ" is translated into co-ordinate form. Co-ordinate geometry of 3 dimensions sounds very imposing, but in fact can be taught to young children of limited ability if it is presented concretely – with vertical straws on a perforated board, or, in an agricultural setting, sticks driven into a muddy piece of ground. Three-dimensional co-ordinates are an effective device for the design of any complicated solid object – an aeroplane, a building, a piece of metal work. At a more advanced level, co-ordinates provide a framework for most physical problems and also as a way of representing purely mathematical ideas; vectors, Hilbert space, etc.

"New" Mathematics: Errors and Possibilities

34. I do not myself accept any meaning of the "new mathematics" or "modern mathematics" other than the research advances since 1900. As commonly misapplied, these phrases are used to give prestige to any teaching innovation, good, bad or indifferent. In U.S.A. "traditional" is used to describe their tradition of very bad, rote teaching and thence (by double speak) to smear any topic previously taught. It is therefore necessary to discuss each innovation individually, and to appraise it as good or bad.

35. Ontario, lying uncomfortably close to U.S.A., has suffered on occasion from an uncritical acceptance of American ideas. An official curriculum introduced a few years ago organised the work of each year around such topics as the set of natural numbers, the set of integers, the set of rationals and so forth. This is an entirely incorrect basis on which to build, reflecting the procedure of the graduate school rather than the needs of the child. In the kind of approach visualized by Miss Biggs (whose prestige, I am glad to say, is steadily growing in Ontario) a young child may meet whole numbers and fractions in his first few encounters with measurement. The abstract approach has driven school mathematics and school science farther apart – the opposite of what is needed both on educational and on

technological grounds. The effects have been most disastrous for the less academic but more active and practically minded child. In a course for such pupils (15 years old) great stress was laid on "the set of irrational numbers". One over-zealous teacher had to be restrained from setting an examination question "Prove that the set of irrational numbers is not closed under addition". I cannot imagine how a teacher would hope to get technical students excited about such a topic which is completely irrelevant to their purposes and interests. What makes such regulations even more tragic is that all the relevant information about irrationals would arise, naturally and incidentally, as a passing comment on Pythagoras Theorem, which is of real and immediate concern to technical classes. The diagonal of a square involves $\sqrt{2}$. A teacher can point out that tables give approximate values of $\sqrt{2}$, but you will not get 2 if you square these, since in fact there is no fraction p/q whose square is exactly 2. The reason might be indicated; squaring causes each prime factor to occur an even number of times, a single factor 2 cannot be obtained.

36. Excessive logical analysis can inhibit mathematical thinking. This was demonstrated long ago on the largest possible scale, that of world history. The following passage is from Cajori, *A History of Mathematics*; the final sentence is quoted by Cajori from the mathematician Hankel; –

> "The Hindus never discerned the dividing line between numbers and magnitudes, set up by the Greeks, which, though the product of a scientific spirit, *greatly retarded the progress of mathematics.* They passed from magnitudes to numbers and from numbers to magnitudes without anticipating that gap which to a sharp discriminating mind exists between the continuous and the discontinuous. Yet by doing so the Indians greatly aided the general progress of mathematics. "Indeed, if one understands by algebra the application of arithmetical operations to complex magnitudes of all sorts, whether rational or irrational numbers or space-magnitudes, then the learned Brahmins of Hindustan are the real inventors of algebra" ".

The Indians, incidentally, were applied mathematicians. They wanted mathematics in order to do astronomy and (like the 17th century discoverers of calculus) did not have the inhibitions of the Greeks, who were not trying to go anywhere.

37. Good mathematics requires a balance between the Indian and the Greek approaches. Good teaching selects whatever mixture of these is most appropriate to the pupil being taught.

38. Now of course the American college professors who created the vogue for the set of irrationals have a perfectly sound point in logic. If for instance we determine $x = \cos 72^0$ by solving the 5th degree equation that expresses $\cos 5A = 1$, we are assuming that the usual algebraic procedures may be applied to the irrational number x. No doubt at some stage, for some students, the implications of this should be analyzed and, so far as possible, justified. What is overlooked is this. For a teacher in a neighbourhood of boisterous children, the first consideration is not the logical precision of his lessons; rather, it is the dramatic impact, how far the pupils will look forward to the lessons and feel that they are learning something exciting and worthwhile. If a course is dull, it does not matter how sound it is in

other respects; pupils will not pay much attention to it. In planning courses much more attention should be given to this question. At frequent intervals it should be clear to the pupil that the last chapter has enabled him to design, or make, or understand, or do something he could not do before. The first thing a teacher must prove to his class is that the course is not a waste of time, and the proof must be a spontaneous reaction in their hearts, not an argument grudgingly accepted by their heads.

39. In passing, I believe an excellent form of teacher training is to go into a street or park or public place where there are children over whom you have no disciplinary powers, and start doing something to see how many children come round you, how long they stay, and what questions they ask.

40. To return to the question of logic; a "preview" approach in mathematics is perfectly justifiable – that is, first to develop a subject informally, in a way that commands the intuitive assent of the learner, and to show what can be done with it; at a later stage, an analysis of the logical foundations may be undertaken. This is generally recognized (within the Commonwealth) in the teaching of calculus; an intuitive treatment should precede a course in analysis. In the same way, I would explain how Euler arrived at the connection between trigonometry and e^{ix} before discussing the Argand diagram or the logical account of complex numbers. First show that a method leads to interesting results, they tidy up the logic. For if it does not lead anywhere, why should the learner spend time on it? Of course it is good to prepare the ground for the later developments by indicating that loose use of infinite series can lead to fallacies, that pictures can deceive, and so on. How much this is done must depend on the teacher's estimate of the class he is teaching.

41. In recent years we have heard a lot about commutative, associative and distributive (C.A.D.) properties. It seems natural to ask – what is the role of these concepts in mathematics? – how did people come to think of them? These terms appeared first in the period 1800-1840, shortly after Gauss, Argand and Wessel had given a geometric interpretation of $\sqrt{-1}$. The entity i had always been something of a puzzle; it was not a number, and yet algebraically it seemed to behave like a number. When addition and multiplication of complex numbers were defined geometrically, the question naturally arose – what properties of these operations must we establish, in order to prove that ordinary algebra works for them? It emerged that most of the things done in algebra were logical consequences of the C.A.D. properties; any system with these properties could be handled *as if* it consisted of numbers. Hamilton, trying to generalize complex numbers, found quaternions with properties A. and D. but not C. Matrices followed soon after.

42. Great mystification is caused if teachers and pupils are told that some concept is important but are not shown significant applications of this. That certainly happened in the U.S.A. where words such as Set and Commutative received a kind of religious veneration. I therefore distinguish between the *unobtrusive mentioning of an idea* (which is an excellent way to prepare pupils for future work) and the *stressing of it,* which indicates that you are about to use it in deriving theorems of some substance.

43. The Americans were, I think, quite correct in holding that, from the very first lesson in arithmetic, we should be preparing children to do algebra one day. The early mention of the C.A.D. properties is in order. Thus we would not merely teach particular facts such as 2 + 3 = 5 but raise questions of more general import, such as "When numbers are added, does the order matter?", "And for multiplication, does order matter?", "What is the craftiest way of working out 58 × 3 + 58 × 7?". A good pupil in a traditional arithmetic class understood all these things; however it is quite sound to make sure they are brought to the attention of every pupil. The ideas involved are evidently helpful in arithmetic and in beginning algebra.

44. The C.A.D. properties can be used to introduce negative numbers, as in Durell, Palmer and Wright's book of 1920 and later in American S.M.S.G. schemes, though I would not regard it as good teaching to rely solely on this approach. Formal properties are singularly unreal to many children, and in fact do not clear up all the logical questions involved. Pictorial and inductive arguments should also be used to establish confidence in the use of negative numbers.

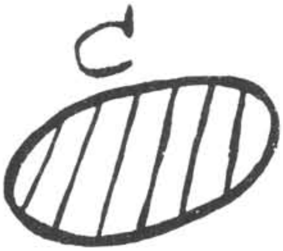
45. C.A.D. most properly takes the centre of the stage when pupils have met and used systems other than numbers (notably matrices and complex numbers) and have met new features, such as, in matrices, quadratics with an infinity of solutions. At this stage the question of which algebraic processes and theorems remain valid arises very naturally, and the justification of algebra by the C.A.D. properties makes sense to learners.

46. I have always thought that the splitting of the "in-any-order rule" into the pair of properties, commutative and associative, is a somewhat subtle affair, and some thought might be given as to how, and at what stage, this might best be done. The qualifications of the teachers are relevant. In some places, teachers are now making incorrect statements about sets where before they only made incorrect statements about numbers.

Modern Mathematics Proper

47. It seems reasonable to suppose that, as time passes, some mathematical results of the present century will find their way into schools. There is a difficulty in recognizing which results these should be. With classical mathematics, we have a fair idea of how each topic relates to other parts of mathematics and to applications. Recent mathematics is split into so many specialities and is so abstractly presented that it is often difficult to realize the inter-relations or to recognize that some paper is helpful for a problem one wants to solve. Associations and institutes should encourage mathematicians and users of mathematics to write understandable accounts of the origins and applications of recent mathematics. To some extent this is already happening.

48. A single example, taken from topology, will perhaps illustrate what I mean. It is a generalization of a classical result, in complex variable. Suppose we have a polynomial $f(z)$ and want to know how many solutions $f(z) = 0$ has inside a curve C. An interesting theorem gives the answer. Suppose that, as z goes round the curve C, the point $f(z)$ goes round the curve K, which, in our figure, makes 2 circuits of the origin. Then we can assert there are 2 zeros (or 1 double zero) of $f(z)$ inside C.



49. Now complex numbers are an extraordinary and unique system. Our theorem seems a very special and limited one. But in fact, by picking out its essential basis, it can be made very general. Suppose we consider not merely the contour C but imagine a membrane covering its interior. This membrane will transform to a membrane with boundary K, and there will be 2 layers of it over the origin. If, instead of requiring f to be a polynomial we simply require f to be continuous, the transformed membrane may have folds in it, but we can still assert there will be *at least* 2 sheets of the membrane over 0, hence 2 zeros of f inside C. This theorem generalizes to higher dimensions, and gives a useful way of locating the solutions of a complicated system of equations. Incidentally this method meets the criterion given by Professor A.J.M. Spencer in his article *The Education of Mathematicians for Industry* (Mathematical Gazette, October 1967); it allows us to find approximate solutions of real problems, rather than neat, exact solutions of unreal problems.

Mathematics and Utility

50. What mathematics do people actually use in life? In principle this ought to be one of the easiest questions to answer, for, unlike most educational questions it does not involve the nature of the human mind, about which we know almost nothing. It is relatively easy to count how many people do a particular job and what mathematics is, could or should be used in such work. Of course there is the difficulty of forecasting future developments. Yet at least we could establish short term trends – automation is wiping out the demand for a certain skill at so many thousand jobs a year, and is creating the demand for some other skill at such and such rate.

51. It is necessary to maintain a mind at once open and sceptical. An educational system is like an army; it takes time to move it from one spot to another, and still longer to bring it back if your first decision was wrong. I have been struck by the loose thinking about education of many, including mathematicians. Someone will say a certain topic is important when all he really knows is that it is much used in his own (non-applicable) research. Someone proposes a topic for the school syllabus on the grounds that it is used in electronic computing, but he omits to mention whether it arises in research, manufacture, maintenance or programming of computers. A mathematician in love with his speciality may claim it is practical because it has been used in a single scientific paper.

52. My other activities have never allowed me to give more than fragmentary attention to surveying the uses of mathematics. And in fact an adequate treatment

requires far more than can be given by one individual. I have suggested on various occasions that the users of mathematics ought to make a periodical report, available to teachers, on trends in the applications of mathematics. Even a small journal, abstracting the most important published information, would be helpful. The reports should make clear whether the developments affected industrial or agricultural countries, many or few workers, scientists or technicians, skilled or unskilled. Vague general statements would be forbidden, and concrete examples would be given of the problems involved. Such a periodical report would be of great value to many people besides teachers, and the Commonwealth probably has the resources to arrange it without setting up any elaborate new organisation.

53. Of course we cannot predict far ahead. Forty years ago (when teachers now retiring were just beginning) scoffing would have greeted a forecast that in 1968 unemployment would have been relieved because many people were gainfully employed trying to put a man on the moon. No doubt 2008 will prove equally unexpected. My proposal is not intended to provide teachers with a crystal ball to see the future; it simply tries to provide them with eyes to see what is happening now, so that we may make an intelligent guess what to expect to-morrow, and keep correcting our guesses at the first opportunity. Needless to say, schools should not try to teach the *details* of to-day's technology (which will soon be out of date) but *general principles* that are likely to endure. However to-day's examples, properly used, give reality and interest to lessons.

54. In 1945-7 the College of Technology in Leicester collected examples of the use of mathematics in the city. Since then of course computers and automation have brought many changes, but one conclusion still stands; the widest use of mathematics is not to solve problems, but as a language in which one learns science and technology. More recently, I made a small sampling of books on various subjects, to see the kind of mathematics used. Books are not always good indicators of new ideas (since authors and readers alike are usually unaware of recent mathematics) but they give some indication which traditional topics retain vitality. Elementary algebra is certainly one of these. It is hard to see how any development in higher mathematics can supersede the use of algebra for stating simple scientific laws, and making deductions by combining such statements. Fluency in reading algebra, the ability to appreciate the meaning of an equation or a graph and to associate it with its applications, is and will remain a most valuable asset, in everything from electronics to ecology.

55. In agricultural countries, interest may centre around the sciences related to biology; in these (as also in many industrial questions) statistics plays a large role. Accordingly, it seems that for biologists, and for others who do not intend to take very much mathematics, an attempt should be made to give familiarity with the binomial coefficients and their role in probability. In view of the normal error curve and the Poisson distribution (both of biological significance) enough calculus to understand e^x seems indicated. To treat e^x by algebra (as in Hall and Knight) is appalling. It is most undesirable to have people working with a symbol like e and having no idea of its meaning or derivation. In *Calculus Made Easy*, Sylvanus P. Thompson slipped e in very early and easily. It would be good if some such simple

treatment of a limited part of calculus could be taught as soon as a pupil had a grip on the basic ideas of algebra; calculus could then be used right through secondary school, for instance whenever a graph had to be sketched. Its ideas would thus become very familiar.

56. The ability to read science is of concern to all citizens, not merely to some employees. There was a scare recently about possible harmful effects of radiation from colour television sets. Here an everyday question involves two profound scientific topics – radiation and Mendelian genetics (probability and binomial coefficients again). In the newspapers there have been discussions of how many parts of sulphur dioxide per million parts of air industry can safely subject city dwellers to; whether the spread of industry will reduce the oxygen content of the air below that necessary to sustain life or, more conservatively, whether the increase of carbon dioxide in the atmosphere will cause the polar caps to melt and raise the sea-level by 180 feet; the poisoning of animals and human beings by the unwise use of pesticides; all kinds of unexpected and undesirable effects of substances sold in chemists' shops; and of course nuclear fusion. Not all the fears may be justified, but the very raising of these questions is a sympton of the enormous increase of man's power to interfere with nature. We are in the position of the sorcerer's apprentice; we have much more power than sense.

57. It may be thought that most people are incapable of appreciating the scientific issues that now permeate our lives. Indeed I do not believe that genetically mankind is any more intelligent than it was half a million years ago. But understanding is not a purely individual achievement. Literacy once meant the ability to master several thousand Chinese characters; to-day it means the ability to learn 26 letters and to cope with some oddities in English spelling, while in Ghana it simply means the power to master a fully phonetic alphabet. An ancient Greek needed something like genius to recognize that the earth was round; to-day a child, too young to go to school, may see on television a picture of the earth taken from space, and grow up never doubting that the earth is round. There are always ways to make ideas clear; our task is to find them. Earlier generations would have been astonished at the idea of every child learning to read. It may not be very long before we take for granted that every child can read mathematics.

REPORT OF WORKING GROUP A.2.

Chairman: B. Noonan, Ph.D. (Canada)

Introduction

58. Because of the great variety of forms of curricula throughout the Commonwealth, for teaching mathematics at secondary level, it was the opinion of this group that its work could best be carried out by initial discussion of detail which would lead to the enunciation of principles and recommendations which would be representative of the wealth of experience of attending delegates. Such principles and recommendations could then serve as guides, with all the reliability that experience, training and considered opinion can give.

59. Secondary education has different meanings in the various countries represented and the effect of, say, an external examination taken at the end of the third year cannot be ignored. Nevertheless, it was felt that the differences resulting from these variations were not so great as to inhibit a general discussion.

The transition from primary school

60. The first problem considered was that of transition from the primary school and delegates discussed the knowledge and skill which one would expect of entrants to the secondary school. It was agreed that:

> Any attempt to over-load the primary school syllabus should be avoided and that, although certain elementary skills were essential, the attitudes with which the child approaches mathematics were all-important. It was stressed that examinations should concern themselves not only with the acquisition of skills but also with the development of understanding and attitude, particularly in those countries where some selection for secondary education has to be made.

Planning a new syllabus

61. The provision of syllabuses in secondary schools was considered. The general view was that:

> It was desirable that teachers should have considerable freedom in deciding syllabus content, although it would be unrealistic to think that all teachers are yet ready for this responsibility. The setting up of Syllabus Committees in countries where they do not exist, should be encouraged. On these committees teachers should play a dominant role, but personnel from industry and Government should be included. In cases where external examinations are set for overseas students, the examining bodies should be guided by local requests.

Organisation

62. The group discussed school and classroom organisation and recommended that:

(a) School principals should arrange timetables to permit setting in mathematics. Under this arrangement a group of classes in a particular year have mathematics simultaneously, the students being assigned to classes according to ability in mathematics. This arrangement allows a student to transfer at any time during the year to a class which accommodates his ability.

(b) Classroom furniture should be chosen so as to facilitate group working and classroom activities which are inhibited by traditional patterns.

(c) Where possible, mathematics teachers should have individual classrooms to which classes may go in turn. Failing this, a minimum requirement is a mathematics room or a laboratory suitably equipped.

The common core

63. The desirability of providing a common core syllabus in the early years of secondary school was considered. It was agreed that:

All children in the first few years of their secondary education should pursue courses with a common core. Those children more able in mathematics should be given the opportunity to study topics in greater depth and to consider additional topics. At the other end of this ability range, topics should be treated in a more informal manner, but at the same time, such children should not be deprived of an enrichment programme.

64. The following principles for the design of a mathematics course were recommended:

(i) The establishment of a natural link with primary school teaching which should make the transition from primary to secondary school as smooth as possible.

(ii) In some situations a practical approach involving student activity would be a useful teaching technique to use. Teachers of mathematics in secondary schools should be familiar with and take an interest in primary school methods and encourage the continuance of these where appropriate.

(iii) In secondary school there should be a shift of emphasis to systematising and formalising, as well as expanding, the concepts and experience of the primary school.

(iv) To make the course relevant to the Community and its mathematical needs as far as they can be identified.

(v) The school curriculum should be integrated so that the contribution of a subject to each of its related subjects can be used to the best advantage e.g. the mathematical concepts necessary to physics should be co-ordinated with the physics programme and, on the other hand, aspects of physics which could serve to illuminate the need for mathematical concepts should be co-ordinated with the mathematics programme.

(vi) Proof or some other form of justification should be given for all formulae, algorithms and theorems considered.

(vii) In cases where the child's formal education is to terminate after 3 years, he must leave school not only with an acceptable degree of numeracy (i.e. insight into mathematical thinking in general and into all basic arithmetical techniques) but also with that flexibility of mind which would enable him to apply himself to problems he will meet as a member of his community.

The first three years

65. In considering the needs of pupils in their first three years of secondary education no attempt was made to lay down a syllabus or to suggest the order in which topics should be taught. The depth of treatment of these items, and the way in which they would be introduced would vary from country to country and indeed from class to class or even pupil to pupil. For example, although all pupils would be expected to be able to solve linear equations, quadratic equations would be included only when there was a clear need. In the opinion of the group, the following topics are those which should be encountered by all students.

Number

number line, extension to negative numbers, rationals, irrationals, and reals

place value, scientific notation (e.g. $2{\cdot}38 \times 10^{-6}$)

computational aids (e.g. the slide rule)

percentage, ratio, approximation and error estimation

Algebra

variable, functions

algebraic expressions and operations

relations, equivalence relations

inequalities, identities, formulae

equations, solution of equations

graphical representation

Geometry

area and volume, the theorem of Pythagoras

similarity, angle measure and angle properties,

elementary trigonometry

plans and elevations

co-ordinate geometry.

Identity elements and inverse elements should be studied in geometrical as well as in numerical and algebraic contexts.

Statistics and Probability

It was suggested that consideration of probability should spring from the pupil's own experiments, and that the statistics studied should be confined to representing data by means of graphs or tabulation, the interpretation of data exhibited in these forms, and the use of the measures, mean, mode and median.

66. Terms such as equivalence relation were not expected to be used by pupils, or necessarily to be defined explicitly, but nevertheless the ideas should permeate the teaching. For example, children should realise the similarity between 'is equal to' 'is parallel to' and 'was born in the same year as'.

67. It must be stressed that this skeletal plan must be supported by a full enrichment programme appropriate to the pupils concerned, for example, matrices might be studied not only for their own sake but also to increase the pupil's understanding of such items as operations, equations, inverse elements and co-ordinate geometry; while the study of transformation or vector geometry would, amongst other things, greatly increase a child's experience of spatial relations. The group felt that the choice of such topics should be left to individual countries or teachers.

Fourth and Fifth Years

68. When considering the syllabus for the fourth and fifth years, the group paid special attention to the fact that for many students this would mark the end of their formal education in mathematics. For such students topics with social and economic relevance should be given particular consideration. The following topics were thought to be appropriate:

(a) Linear programming
(b) Statistics up to sample theory and significance (including projects of an experimental nature)
(c) Co-ordinate geometry and vectors
(d) Operatives in systems other than the real numbers e.g. matrices, geometrical transformations
(e) Computing, flow charts, programming in some sub-sets of high level language, Computer appreciation
(f) Social arithmetic e.g. income-tax, hire-purchase etc.

Sixth and Seventh Years

69. The choice of topics at this level should be sufficiently wide to satisfy the needs of:

(i) those whose formal education in mathematics will terminate at the end of this stage;
(ii) future social and biological scientists;
(iii) future physical scientists and engineers;
(iv) future specialists in mathematics.

70. There was considerable discussion of the content of modern sixth-form syllabuses. It was agreed that some of the principles which should guide the choice of content are:

(i) the need to systematise parts of mathematics, for instance, co-ordinate geometry, properties of polynomials, elementary number theory;
(ii) the need for refining the ideas of proof and the use of axiomatic processes where possible. For example, elementary abstract algebra can be used to

justify rigorously results presented informally in the earlier grades. On the other hand, the introduction to the calculus should be based on an intuitive approach to limits;

(iii) the need to make mathematical models of physical situations: i.e. abstracting from a physical situation by symbolising quantities and relations, and arriving at mathematical conclusions in the model which can be interpreted in terms of the physical situation. Such illustrations are to be found in mechanics and probability;

(iv) the need to develop facility in the use of mathematical methods.

71. It is recommended that students should be familiar with some of the precise language of mathematics, including contemporary terminology relevant to the course they are pursuing.

PLENARY DISCUSSION OF REPORT OF WORKING GROUP A.2

Survey and Comments

72. In the organisation of the Conference, the Planning Committee, when discussing secondary education experienced difficulty even when only some six countries were represented. In the range covered by the Conference, the difficulties were magnified in that there was a wider spread of aims, objectives and degree of selection for secondary education. In the United Kingdom, for example, there was secondary education for all pupils up to the age of fifteen years (soon to be sixteen years). Such a non-selective pattern did not apply to most of the developing countries, but some delegates reported experiences with various versions of secondary education such as vocational schools, secondary modern schools and community colleges for some pupils in addition to the strictly academic schools of the "grammar school" type. In most of the countries represented, secondary education was highly selective and was influenced by the requirements of a strictly academic School Certificate, GCE "O" or "A" level, as the case may be.

73. Evidence as to the climate of thought and consequent action was given by delegates from a number of the developing countries when they said:-

(i) "I think the kind of secondary education that is given in developing countries is always geared towards examinations" ... "If you look up the advertisements you will see that a School Certificate is always required and the children in the secondary schools are just aiming for this".

(ii) "The 'secondary modern' schools answered our problems in terms of keeping children off the streets, but then the children could not get jobs after completing the course, so these schools are no longer popular ... the grammar school still holds the supreme spot".

(iii) "... we have a new trade school for children who have ended their education at the age of fourteen. The objective of the trade school is to provide technical education to the age of sixteen years for the skilled labour my country needs. There will be a further development beyond the age of sixteen in a community college. What mathematics is to be taught at these two places is a big question".

(iv) "We have found that parents understand secondary education to mean the traditional form of education which leads to what is sometimes described as a "white collar job". There are not enough jobs of this kind; we need technicians ... and so our Minister directed that the basic education should be broader by the inclusion of such subjects as woodwork, metalwork, art and music".

74. This delegate then pointed out that such pupils often went on to a technical school for vocational training and they were often placed in jobs long before the end of their course.

75. In reply to a suggestion that any marked expansion of secondary education as currently practised in many countries would not of necessity be a good thing, it was thought that more secondary education of a different kind is necessary but that the whole problem is linked with the questions of teacher supply and training, of resources for education generally and the economic position of the country concerned.

76. The advice and recommendations of the report, being couched in rather general terms, places the duty of implementation with the country concerned according to its particular needs and problems. No specific attempt was made to lay down a working syllabus, or to order the items listed; still less to deal with detailed teaching techniques. The outcomes would be different in different countries. Such a possibility of variety was welcomed as being a desirable thing.

77. *A common core for the first three years of secondary education* received considerable support. Delegates felt that the common core outlined in the report could well serve the purposes both of the academic pupils and of the remainder who had not the same academic aspirations – and although it represented a desirable minimum it should be interpreted and enlarged by an enrichment programme designed to meet the needs of the pupils concerned. A note of caution, however, was sounded when it was pointed out that the crucial matter was the manner in which the topics were to be introduced, discussed and handled in the classroom. The existence of a list of desirable topics would not of itself ensure the interest and involvement of many of the pupils.

78. Any *common core,* since it represents the skeleton of a desirable minimum, can be criticised as unimaginative and lacking in bias towards the needs of any particular community. The point at issue is surely whether or not such a course leads the pupils to a sound grasp of the basic ideas and structures of mathematics; for it is on these that any vocationally biassed additions can be subsequently made. Furthermore, it is these basic ideas, provided that they have been suitably presented, which will enable the pupil to have a desire to continue learning mathematics and to find pleasure and profit in so doing.

79. Since in many of the developing countries, agriculture in one form or another was a staple industry, the question was raised as to the part a mathematical education could play in these circumstances. Whilst at present many of the agricultural workers were illiterate, or perhaps with a primary school education at most, this does not lessen the duty and responsibility of the authorities to provide these workers with a mathematical education. To quote a delegate "... even on a farm one must know some elementary mathematics, one has to make decisions, decisions for example as to what size of chickens to sell". The speaker claimed that a mathematical education contributed to this, and he advocated a continuance of the kind of approach given in the Primary School Report Group, as being suitable for such secondary pupils.

The limitations of a specific "agricultural mathematics" course at the secondary level were mentioned. Some countries had rejected such courses on the grounds that they did not provide for their growing needs and aspirations for wider technological development. Thoughts of such a restricted mathematical curriculum did not prove acceptable to the conference, and so further support was given to a "common core".

80. Since this Secondary Report suggested *forms of specialist mathematics rooms,* and the Primary Report (A.1) made no such suggestions, the distinction was questioned. It was pointed out that in a primary school the usual pattern was for one teacher to take one class for all the time in one place, and that in such circumstances it was easier to draw materials from a store and feed them, possibly by a trolley to the required place at the required time. In a secondary school staffed by specialist or semi-specialist mathematicians, the case for mathematics rooms or centres was considered a stronger one.

81. The recommendations of the Report relating to facilities for mathematics learning found general acceptance, as did the suggestions for suitable approaches to be used, and patterns of school organisation thought to be appropriate. Sporadic references continued to emphasise "practical approach", "continuing the experimental work begun in the primary schools" and the continuance of science and mathematics being studied together. The philosophy and the approaches found in the Primary School Report should extend to the secondary schools to a greater measure than at the present time.

82. The absence in the Report of any reference to the *history of mathematics* was deplored by one delegate. The meeting felt that a serious study of mathematical history was more appropriate to older students say at a Teacher Training College; nevertheless as one delegate said "perhaps every child should be made aware of how mathematics had developed and is developing: this is just a part of good teaching". Another delegate said that there was a danger of the history of mathematics deteriorating into a list of names and dates, yet it was important to include a well-devised course because many people appear to believe that the body of knowledge they acquire always existed and that they do not realise that there were times when it did not exist.

83. *Patterns of collaboration with industrialists and technologists* were discussed and served to highlight the necessity in some regions for forms of secondary education other than the strictly academic. In the words of a delegate "sitting around a table

with a number of industrialists writing up a school syllabus does not seem to be a practical proposition". Whilst this was agreed, it was felt that the users of mathematics had pertinent views to express, and members from widely different areas outlined patterns of operation which had been, or were to be, used. To take some examples:-

(i) In an African country a seminar lasting a week or so in which representatives of industry and of education agreed on desirable aims. Having established the objectives, it was left to the teachers to design a syllabus and introduce the content into the schools.

(ii) In a West Indian country a National Council exists for the control of technical examinations. This was stated to work fairly satisfactorily at the craftsman's level but that it was beset with problems at the technician's level, in that the courses at that level tended to correspond with GCE "O" and "A" levels. A student who successfully passed these, felt himself to be a potential university entrant, and so many of the students joined academic university courses and thus dropped a technological course.

(iii) Note was taken of the existence of an English project, *Mathematics in Education and Industry*. This project was based on the work of the Industrial Committee of the Mathematical Association: it seeks to effect liaison between industry and education and to develop syllabuses in the light of experiences gained. Revised GCE examination syllabuses have been prepared. Further information can be obtained from The Mathematical Association, 22 Bloomsbury Square, London, W.C.1.

For those wishing to make fuller study of vocational education a suitable source book would be:-

Vocational and Technical Education by Hugh Warren, Unesco (1967). This is a comparative study of present practice and future trends in ten countries (U.S.A., U.S.S.R., and eight European countries).

Another form of collaboration exemplified was where the representatives of industry came into the schools and colleges and became part time instructors in their particular specialism. In some regions, it was reported, the students go into industry for their practical training.

Vocational and technological education were important growing points for all countries, but no matter what form they take, they can only be based on sound mathematical learning in the primary and early secondary years.

84. *The relationship between the curriculum and the examination* was thought to be important. It was stressed that it was necessary that the examination should follow the curriculum and not, as so often happens at present, the curriculum be exclusively based on an examination syllabus. The setting up in England and Wales of the Schools Council for Curriculum and Examinations was instanced as a significant attempt to link these aspects of education. "It makes us, when we think about curricula, think about examinations and vice versa", said one delegate.

Delegates were very interested to hear of the objectives of CREDO (Curriculum Renewal and Educational Development Overseas), particularly that it sought to support rather than to initiate curriculum development.

85. The view was expressed that the rather cautious wording of the Report failed generally to communicate the thrill and sense of excitement of those teachers and pupils who are involved in new mathematical developments. This feeling of adventure was evident from many of the countries which had made a start on a new line. It was hoped that teachers and authorities would use the freedom offered by the report to experiment both with approach and with content, bearing in mind the needs of their country and the power of mathematics not only as an intellectual exercise, but also as an instrument for the general education of the person. In this respect it is hoped that the report will act as a broad guide. The content of the "common core", if treated suitably, could be regarded as the means of developing in pupils the power to discover and master problem situations together with the techniques and skills arising from them. At the same time it forms a broad basis on which further programmes can be built according to the needs of a particular country or of a particular situation. The Report should not then be taken as a policy of despair, but as an expression of optimistic hope capable of realisation.