CHAPTER IV

The Teaching of Mathematics at Primary Level

Lead paper by Miss E.E. Biggs, H.M. Inspector of Schools,
Department of Education and Science, London.

Introduction

* *"In mathematics there's always a pattern –
you've only got to look for it"*

1. Mathematics is an abstract subject, and because of this, in the past, it induced fear in both young children and students. But in the last ten years there has been a fundamental change of outlook and the new generation is no longer condemned to a complete diet of instruction by teachers (however good). Nowadays, pupils are encouraged far more to think for themselves and they investigate mathematical problems in individual and sometimes highly original ways. The ethos of this new era is embodied in Professor Polya's words: "Abstractions are important; use all means to make them tangible. Nothing is too good or too bad, too poetical or too trivial to clarify your abstractions". The approach to mathematics, the way we present this subject to our pupils of all ages (including university students, as Professor Polya has so successfully shown us) is all-important. For this reason I shall consider approach first and leave content until later.

2. Here my main concern is with children between the ages of 5 and 12. For the more fortunate children of pre-school age, learning has proceeded naturally without effort or interruption. The newer methods in the infant school (5 to 7 years) begin in the same way with the direct impact of the environment and the child's own response to it. This permissive way of working, with children in small groups, was relatively easy to introduce in British schools since syllabuses are not externally imposed and head teachers can devise their own schemes of work and classroom methods.

Historical Background of Changes in Britain

3. The changes in classroom procedure in our infant schools began more than thirty years ago. Teachers concerned themselves first with the quality and scope of the materials they provided (often the normal materials of the environment such as clay, water, sand, scraps of textiles, leaves and shells) and secondly with the learning situations themselves. The quality of children's art and craft, movement and writing became more individual and imaginative; the narrow number syllabus was extended to include purposeful counting and measuring experiences. But until twenty years ago there was a marked contrast between the teaching methods of infant schools and

*See page 50

those of junior schools. At that time junior schools were influenced by examinations at the age of eleven for allocation to grammar (academic) schools. In these examinations the emphasis was on grammar and comprehension in English and on speed and accuracy of computation in arithmetic. It is not surprising that the first curricular experiments in junior schools were in art and craft and physical education, subjects which were not examined. However, in due course the emphasis in English, too, was gradually shifted from formal exercises to creative and imaginative writing with no loss of accuracy and much gain in fluency and vocabulary by the age of ten.

4. Experiments in the teaching of mathematics began about ten years ago. These changes were all the more far-reaching because they were brought about not as a result of external pressure from Universities or Education Authorities, but by the pioneer work of teachers in their own classrooms. They were part of a wider movement which had already affected many other aspects of the curriculum. Mathematics and science were no longer isolated; they now made a vital contribution to learning as a whole.

Discovery Learning in Mathematics

5. Let us take a closer look at "discovery" learning as this affects mathematics. "In some ways it resembles the best modern university practice".[1] I should be happy to think that most universities followed this practice! This is what Piaget has to say about it: "The goal in education is not to increase the amount of knowledge but to create opportunities for a child to invent and discover. Teaching means creating situations where structures can be discovered". When we say that initial curiosity is often stimulated by the environment the teacher provides, we are admitting that the teacher selects and structures the programme. Sometimes a worthwhile piece of mathematics will be initiated by a child with a special interest. More usually the environment and the teacher's questions will catch the child's imagination and sustain his enthusiasm. But the permissive classroom in which children are working in small groups, often on quite different problems, makes heavy demands on teachers and especially on inexperienced teachers. Both children and teachers have to accept considerable responsibility. Many teachers make a gradual transition. Most of the class work from a textbook while one group is provided with new materials; the teacher observes the children and asks questions. From month to month she increases the number of children working with materials until at last the whole class is involved.

6. What are the aims of "discovery" learning? First, to set children free to think for themselves. Secondly, to give them opportunities to discover the order and pattern which is the very essence of mathematics and which is to be found in the natural as well as in the man-made world. Thirdly, to give children the skills. Here I want to put a personal and perhaps extreme point of view. I believe that with experience teachers can plan each child's learning so that he discovers himself all the mathematics we want him to learn, even the skills. Our success in this will depend not only on the experiences we provide but also on the questions we ask. I also beleive that it is just as important for children to be encouraged to devise their own methods in

[1] "Children and their Primary Schools". H.M.S.O. 1967

written calculations as it is for them to make discoveries with shapes and patterns. Later in this paper you will see how children learn from one another to refine their methods.

7. I am convinced, from my own experience gained from working with children and teachers in many parts of the word, that the approach I am describing is not for "enrichment" only. It is an approach which can and should be used in every aspect of mathematics, at every stage and with children of all abilities. This does not mean that every child requires the same kind of experience – far from it. There are those who need little experience with real materials and who enjoy working abstract problems which develop from brief initial experiences. Other children require a lengthy period with real materials before they are able to abstract a single idea or concept. There are some children who will respond to open questions, and others who require more direction. Nor does discovery learning always require the use of concrete materials; it is the form of question we ask which determines whether we are challenging a child to think for himself – or telling him the answer. Dr. Schweitzer wrote: "Only those who respect the personality of others can be of real use to them". We must respect the independent personality of each child, and encourage him so that he takes the final step, however small, for himself. It is this independent discovery which gives him the satisfaction of success – and which requires all our skill to engineer. If we find ourselves telling the answer it is either because the child is not ready for the experience or because we have not planned the work to suit the ability of that particular child.

8. I should like to quote one example from an African school in Rhodesia because I believe that discovery methods would be as useful in developing countries as in Britain. Here I found that teaching methods were almost entirely instructional. I introduced simple materials, for example, congruent boxes, squared paper and string. At first children and teachers found it very strange that these should be used to make mathematical discoveries. Later, however, with adequate experience they became quick to see the possibilities, particularly the younger children.

9. Older students found no difficulty in making discoveries with number patterns – indeed, they had a strong sense of pattern. We had been working with the pattern of the 9 times tables. I wrote: 3, 12, 21, ... on the blackboard and asked them to continue the sequence and to construct similar sequences. They soon discovered that the final sum of the digits was always three. I then wrote several large numbers on the board and asked them to test these for divisibility by 9. "But, madam, they all 'refuse' ", they said with some excitement and were quick to discover why. I see no reason why developing countries should not be able to take advantage of these methods.

10. Because the form of the questions we ask – open-ended or directed – is so important, I should like to give you two contrasting examples:-

For several years I had given considerable direction to children and teachers in isolating the variables when working with a pendulum. For example, one assignment was: "Time the pendulum for 30 swings for lengths 6", 12", 18" as far as 48". Draw a graph. Can you find from your graph the length of a pendulum *which beats seconds?" A group of ten year olds performed the experiment carefully and obtained a reasonable result. A month later I discussed the pendulum with the same group. "What did you discover?", I asked. Their recollections were so hazy and confused that we had to start at the beginning once more.*

Last summer I was working with a group of nine and ten year old boys in downtown New York. We had decided to time various objects which the boys had chosen to roll down a long slope in the corridor. We had no stop watch, so I had added a length of fine string and a piece of plasticine to our collection. When the boys asked for a stop watch, I asked them if they could devise a means of timing from the materials I had provided. On seeing these Richard immediately suggested making a pendulum. It so happened that there was a large hook fixed at a height of 7 feet above the slope, to which the boys attached the longest pendulum they could make. "Where shall we start?", I asked. "Up at the ceiling, straight out", Earl replied. "Does it matter where we start?", I questioned. They decided that it did matter and we set the pendulum swinging. "Does it beat regularly?", I asked. "Let's count", they replied. But the boys decided that the pendulum was swinging too slowly for effective timing. "How shall we change the beat?", I asked. "Shorten the string", suggested Richard. "Lengthen the string", said Adrian. "Add more plasticine", said Robert. They experimented with different lengths until they were satisfied with the beat. "How shall we count?", I asked. "Number of swings in half a minute", said Mervyn. I gave him my watch with a second hand and he was soon timing confidently from any starting point of the second hand. "It swings much faster when we shorten the string", they commented as they continued their experiments with different lengths, "but it doesn't change when we alter the weight". These experiments were rough and ready and needed a careful follow-up with a good point of suspension, but I want to stress that all the thinking had been done by the boys themselves and the suggestions came from them and not from me. "The sense of personal discovery influenced the intensity of their experience and vividness of their memory".[1] We were so engrossed that we forgot all about the purpose of the timing!

But the initial impulse needs to arise from some experience in which the children are interested. *For example, the assignment: "Find the length and breadth of this room", may seem pointless to children, especially if they have done this already the previous year. But some nine year olds were enthusiastic when the question was worded: "Do you think this room is twice as long as it is wide? In how many ways can you find out without actually measuring?" One girl counted the bricks round the wall, a second counted the square tiles on the floor and a third counted the ceiling tiles. The others decided to pace the width and the length. They were at first puzzled and then amused when they discovered that each took a different number of paces. This investigation led to a discussion of 'normal' and 'average' pace, approximation and ratio. It also resulted in several oral calculations and was in all a most worthwhile investigation.*

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[1] op.cit.

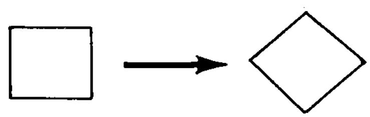
Discovery Learning in Arithmetic

11. I have said that it is important that children should be given the opportunity to devise their own methods of written calculation when they can no longer cope with the complexities in their heads. Let me explain how this can be done. First we must give young children (between the ages of 5 and 8) preliminary experiences which will enable them to learn what arithmetic is about. These experiences should include:

(a) Matching sets (one-one correspondence): setting a table for a meal, straws to milk containers, spoons to bowls, chairs to children.

(b) Counting, matching a number name to each object. Cardinal numbers. Numbers in sequence; the number line.

(c) Measuring; length, weight, capacity, time, area, using simple instruments. Learning to measure is a lengthy process involving several stages. For example, in considering length a child at one stage thinks that a pencil changes its length when it is placed in different positions. Eight and nine year olds who have not had the necessary experience say that a square geoboard turns into a diamond of different shape when it is rotated into a new position.



It is only by means of sufficient first hand experience that children come to understand that the length of lines does not change when their position is changed. In children's first experience of measuring they do not realise the need for equal units.

A class of eight year olds in Ontario had had their first experience of measuring. They told me excitedly, that the width of the room was 6½ bodies. "Whose body?", I asked. Three friends stood up, two of the same height and one, Roger, a head taller than the other two. "Would it have made a difference if you had used Roger only?", I questioned. "Yes, he would have got tired", was the amusing reply, which gave me the clue to their method. "Would your answer have been 6½ Rogers?", I asked. After some thought a girl said there would have been fewer Rogers, because he was taller. But the others maintained that the answer would be the same because it was the same room! It is only by means of extensive measuring experiences that children develop understanding and confidence in all aspects of measuring.

(d) Operations of addition, subtraction, multiplication and division with number, length, weight, capacity, money, time.

It is interesting to note that, in the real environment, subtraction and both aspects of division usually precede addition and multiplication.

For example, using two ribbons of different lengths, children first notice that one is longer than the other, and eventually may measure the difference (by complementary addition). One child asked, "How many ribbons of the short length can I cut from the long piece?" (the subtracting or grouping *aspect of division). Another child halved and quartered the long piece (the sharing aspect of division).*

12. What knowledge do children need of number relationships, including multiplication tables? If we believe that children should be able to perform reasonable calculations efficiently, they need to memorise (a) the addition and subtraction number trios, e.g. 3+5 = 8, 5+3 = 8, 8–5 = 3, 8–3 = 5; (b) the multiplication and division number trios, e.g., 6×8 = 48, 8×6 = 48, 48÷6 = 8, 48÷8 = 6. Once children have discovered the commutative laws of addition and multiplication, memorisation of these number relationships is halved. When they try to see whether subtraction and division are reversible (or commutative) they can be encouraged to extend their number knowledge to include the negative integers, using an extended number line, and fractions (or rational numbers). But we cannot escape this necessity for children to memorise the number relationships; first and foremost the addition and subtraction facts up to 20, then the extension to these facts to 100. For this extension a number line (100 units long) and strips (1 to 10 units long) are extremely helpful if children are to discover the repetitive patterns e.g., 9+7, 19+7, 89+7, and also 96–7, 86–7, etc. and also 96–9, 86–9, etc. These patterns must be known before children begin written calculations.

13. When children have this knowledge of number, what experience do we need to give them which will encourage them to devise their own methods of long multiplication and long division? Here, once more, are two contrasting examples:-

The first is of a group of 'disadvantaged' ten year old girls in California. They had estimated the number of pieces of cereal in a jar and were checking their estimates by filling identical mugs until the jar was empty. The first mug contained 59 pieces and there were 40 mugs in all. The girls had no idea how to perform the calculation, until one girl made an attempt to write forty 59s on the blackboard and add them up. The bell released her! Next day the teacher gave the girls six multiplication examples to do, one of which was 40×59. Each girl performed this successfully. These girls knew how to perform mechanical multiplication examples but had no idea of circumstances in which multiplication would arise.

My second example is of a seven year old class in Ontario. Their teacher had experimented for the first time in giving her class first hand experience with volume. They had made a most varied collection of containers. I picked up a bag of macaroni and asked them to guess the number of pieces in the bag. We all made wild guesses. I asked if anyone could suggest how we could check our guesses without counting every piece. "Count in twos or tens" was the first idea. A girl then said that if we could halve the bag, we could count one half and double the number, I showed the class a small cup and asked if this would help. Immediately I received two suggestions from boys. One said, "Fill the cup and count the pieces. Fill the cup until there is no more left and count the cups full". As soon as I left the teacher let the children try this suggestion. The cup held 110 pieces and there were 6½ cups. She wondered whether the children could solve the problem with this information since they had never tackled figures of this magnitude before. "Put two together and that makes 220", said one. "220, 440, 660", they counted. "I know half of 100 is 50 and half of 10 is 5, that makes 55. That's 715 altogether", said *another. The teacher was surprised at the confidence the children showed in attempting a problem entirely new to them. She realised that these children had encountered a situation in which multiplication was required. But they would need further experiments to prepare them for recording their own methods; gradually they would refine these as they had more experience.*

14. It is important to realise that mathematical symbols should be introduced *after* varied experiments. The sequence of events is: experience, discussion, recording in the child's own words, introduction of mathematical symbols as a shorthand form, written calculations devised by the child himself, (and gradually refined, with guidance from the teacher), practice as necessary. It has been interesting to notice that children who are being educated in this way require far less practice to maintain efficiency than children brought up on traditional methods. One period (40 minutes) a fortnight is often found to be sufficient.

15. Here is an example showing how children first devise and then refine methods of long division when the need arises:-

A group of eight year olds had been working with a calendar. How many weeks in 50 days? 80 days? 110 days? asked the teacher. One hundred and ten days drove them to paper and pencil. Each child produced a different method and, when all had finished, methods were compared. The most rudimentary method was:-

days			*days*	
110			*110*	
7	*1 week*	*The most efficient was*	*70*	*10 weeks*
103			*40*	
7	*1 week*		*35*	*5 weeks*
	etc.		*5 days*	*15 weeks*

All the children were attracted to the most efficient method and began to adopt it. One by one they refined the method still further, showing that they understood what they were doing and appreciated the need for efficiency.

16. I could quote many more examples of children's discoveries in measuring, fractions, and multibase arithmetic, but space prevents this. I should be delighted to expand these ideas in working sessions to anyone who is interested.

17. What is the place of commercial structural material in the learning of mathematics? I believe that children should first have extensive experience of the environment, but when, for example, they are beginning to organise their basic number facts, or at another stage their knowledge of fractions, the introduction of structural material will help them to do this. For example, when children can deal with simple fractions in practical situations and can find one quarter (and later three quarters) of varied materials such as a length of ribbon, a bowl of rice, a jug of water and a sum of money, they are ready to use structural material to summarise and gain further insight into their experiences. Later still they will meet fractions as rational numbers on the number line.

18. Learning by investigational methods has not led to any loss of efficiency in computation. The first schools in Britain to adopt the new methods were in urban

areas where classes were of over 40 and a selection test at 11+ was in operation. These schools kept careful records of the results at 11. It was found that in the first year there was no change. After this, year by year, the percentage of children qualifying for places at grammar schools increased until, in one school, it was 50 per cent. This local education authority has now abolished the 11+ examination! Of course the schools gave the children *some* written computational practice (one period every two weeks) but this was a considerable reduction on former methods, and yet the standard of computation had improved.

Mathematical Content

19. Mathematical content, has two aspects: mathematics in the classroom and the mathematical background which primary teachers require if they are to make the most of the situations which arise and are to give children the help they require.

20. There are certain topics which will arise in the classroom whether we plan these or not. I refer to statistics, three-dimensional and two-dimensional shapes, symmetry, similarity and limits. In children's attempts at communication they will use language (oral or written), tabular forms, diagrams, including mapping, pictorial representation of various kinds including block and column graphs, and line (relationship) graphs. The introduction of these topics requires and extensive knowledge of mathematics on the part of the teacher. I shall not attempt to develop each topic in detail; that is the task of the working parties. I shall quote one or two examples of children's work to illustrate children's difficulties and potentialities and to show how number relationships and spatial relationships are interrelated and reinforce each other:-

(a) Volume

The first is an investigation by some six year olds who had made a collection of small rectangular boxes. Their teacher asked if they could find the box which held most. After some discussion the children filled each box with sand and weighed box and sand. They set out the boxes in order but sand ran out at the corners, so they decided to fill the boxes with something else. This was not so easy and it was some time before a boy found that a cubical bead of a certain size would fit into all the boxes. The cubes were then taken out and arranged in a column above each box. At this stage no child noticed that the volumes could not be compared without counting because the starting points were different. Fortunately the classroom was overcrowded and a child bumped into the table and disarranged the cubes. This gave a boy an idea. He took a large sheet of ½" squared paper, arranged the boxes in order of volume, and coloured in a column of squares for each box, matching one square to each cube. This time he started, on his own initiative, from a common base line so that comparison was easy. "Can you see," he wrote, "that boxes 7 and 8 hold the same number of cubes? But they are very different shapes". This comment started a further investigation. Here we see the process of abstraction from boxes filled with sand (and weighed) to cubes, symbolising the volume, and finally to squares, by a matching process.

(b) Shapes

This example illustrates another important point. We live in a three-dimensional world, and two-dimensional shapes are abstractions from the real world. Therefore we should give children abundant experience with three-dimensional shapes before we expect them to abstract the properties of two-dimensional shapes such as squares, rectangles, triangles and circles. Before a child can recognise a square he needs to be able to sort a set of cubes from a set of cuboids and to put into words what he has done. There is no reason, of course, why we should not use two- and three-dimensional shapes simultaneously, as the following example shows:

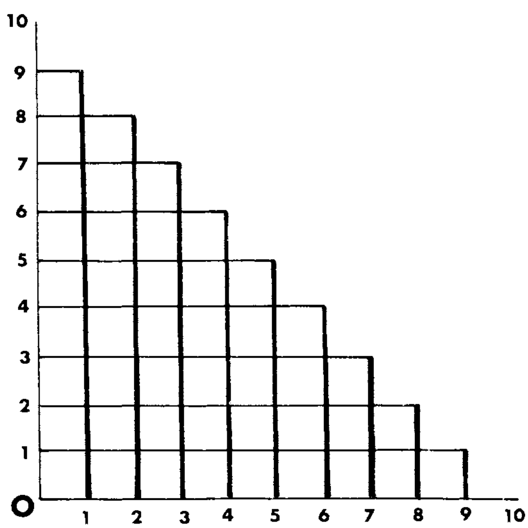
Some deaf five year olds had made a collection of containers of very varied shapes. Their teacher drew a square, a rectangle, and a circle on the floor and watched to see what the children would do. They sorted cubes into the square, cuboids into the rectangle, and cylinders into the circle. It took one girl some time to decide whether a packet of tea with two square ends should go inside the square or the rectangle. Eventually she decided on the rectangle because the packet had four rectangular faces and only two square faces.

(c) Transformations, Spatial and Number Patterns

Most of the spatial discoveries of young children are made as a result of movement and transformations. Pattern has a strong appeal to them.

I asked some nine and ten year olds in Philadelphia to make themselves a halo with a strip of coloured paper and then to experiment with this to find out whether they could change the area enclosed. They experimented with many shapes before they decided, by counting squares, that the circle was the largest shape. At the other extremity, a nine year old boy said he could squash out all the area! I then directed their attention to rectangles and asked them to draw me the complete sequence of rectangles whose perimeter was 20 units and whose sides were in whole units. They first drew these on the

FIG.1



floor in any order. I next asked them to cut the rectangles from 1" squared paper, to put these in order of width, and to mount the sequence on a piece of coloured paper. The paper was small so the children had to overlap the rectangles – and they were delighted to discover the "staircase" which a ten year old had predicted (fig.1)

I asked them to make tables of the width and length and area to see if they could discover why the rectangles formed a staircase. They soon discovered the pattern of the set of ordered pairs (W,L), W + L = 10, although they did not at first arrange these in order of width. When they came to make a table to show the patterns they decided that there must be a pattern in the area column because both the width and length columns showed definite patterns (fig.2).

FIG.2

W IN.	L IN.	A SQ. IN.
0	10	0
1	9	9
2	8	16
3	7	21

9
7
5

Excitement ran high when they discovered the odd number difference of the areas. I followed this by suggesting that the children cut out the sequence of squares with integral sides. The squares were mounted in various patterns. The perimeter and area patterns were recognised as soon as these were arranged in sequence (fig.3). Zero values were added later.

FIG.3

Edge	0	1	2	3	4
P	0	4	8	12	16
A	0	1	4	9	16

When, after some discussion, continuous graphs were drawn, the children's immediate reaction to the area/width graph of rectangles of constant perimeter and the area/width graph of the squares was to ask: "Where is the other half of the squares graph?" I suggested that they should count backwards and see.

Later I asked the children to reverse the constant-perimeter problem. "Fix the number of squares", they replied. I suggested that each child should choose a number of squares and make as many rectangles with these as possible. Once more they decided to use the square tiles on the floor and to *draw round these. But a boy who chose 5 squares found my suggestion too limiting and began experimenting on his own. He showed me the largest perimeter ("with squares as spread out as possible") (fig.4) and the smallest ("squares as close together as possible") (fig.5):-*

FIG.4

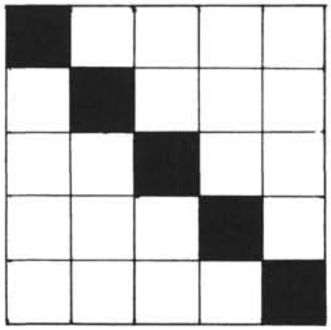


FIG.5

and then set out to "fill" the interval between 10 and 20 units with as many different perimeters as he could. We all became fascinated by this new problem and forgot our rectangles. Some time later I gave the boy's problem to a group of teachers in Ontario. Two teachers from secondary schools spent a morning on this problem, investigating all the possibilities, and discovered a group structure. They found this problem as intriguing as did the boy who invented it. This example emphasises another important point: open situations provide more possibilities for discovery than closed situations.

It was some time before we returned to our original problem: the pattern of rectangles of constant area (12 squares, 16 squares, 36 squares were eventually chosen by the children). This time the relationship of corresponding dimensions (3, 4), (2, 6), (12, 1) etc. was discovered before successive rectangles were cut out and arranged in a pattern. This constant product pattern $W \times L = 12$*, and later the continuous graph, delighted the children because the graph was so different from the constant-sum graph.*

(d) Symmetry

Many normal classroom activities (paper folding and cutting, painting and ink blots, tracings) lead children to make symmetrical shapes and to discover their properties but unless teachers know something of the mathematical significance of symmetry, they may miss opportunities for developing this. Children recognise symmetry at an early age.

Stephen, just five, was a very shy boy. His favourite occupation was painting. One day he dropped a splash of paint on his large sheet of paper just as he was about to paint a picture. He folded the paper to get it into the waste paper basket, and then it fell open. He was so excited by the pattern he saw that he took it to show his teacher. "Stephen's patterns" began to cover the walls of the school. Children's collections included leaves and flowers "like Stephen's pattern", and other flowers showing a different kind of symmetry.

A six year old had been struggling to fit the lid on a nearly square biscuit tin. As he came away his teacher asked him: "In how many ways can you fit *the lid on the biscuit box?" The boy turned to find out, and then, without further experience, replied, "Two ways it will, two ways it won't". That teacher realised the mathematical potential of the boy's struggles with the box and lid.*

A large class of ten year olds had been experimenting with regular symmetrical shapes. They discovered that the angles of a regular triangle were 60°; they calculated the angles of a hexagon (made with regular triangles) at 120°. "Three sides 60°, six sides 120°". The symmetry of this appealed to them. "Is this a pattern?", they asked. This sparked off an enquiry concerning the relationship between the number of sides a regular shape has, and its angle. They plotted a graph showing how the angle changed with the number of sides. The line joining the discrete points was in a curve (fig.6), so the children decided to look for a turning point.

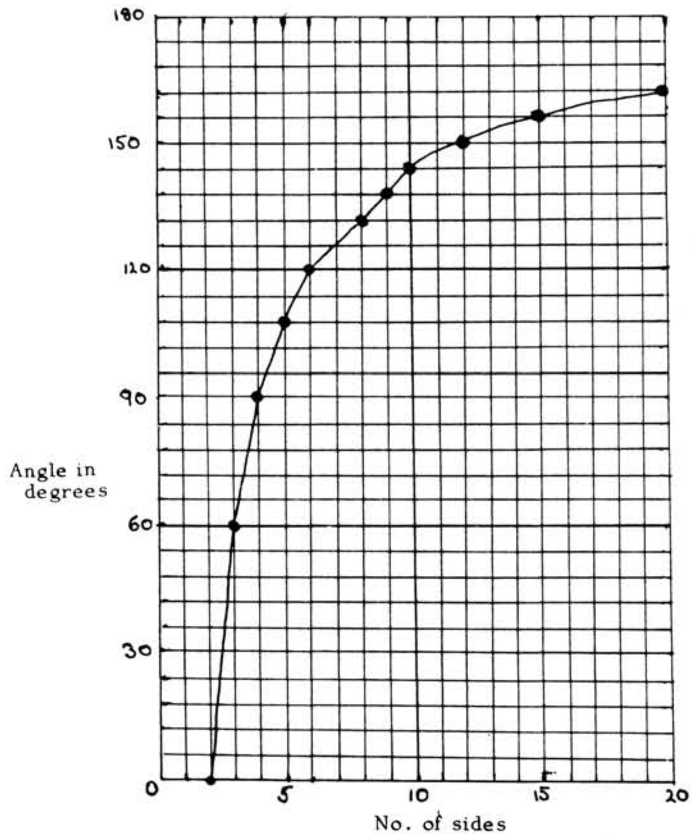


FIG.6

Their teacher was absent for several days, so they worked without interruption or supervision and plotted the graph to 240 sides. The line joining the points showed no sign of turning but they continued, calculating angles of sequences of regular polygons. The last entry was 450 million sides. "When we saw the pattern, we realised we never should reach 180° or turn the graph", they wrote. Throughout, these children were aware of pattern and symmetry in shape and in number.

(e) Similarity

I want to tell you one example of similarity from many, because this illustrates how easily mathematics can arise in other aspects of the curriculum.

A class of ten year olds who kept many classroom pets had noticed that the mouse seemed to feed for longer periods then any other animal. The teacher suggested that they should find out how much the mouse ate. So the children kept careful records daily for a month and discovered that their one-ounce mouse ate, on an average, half an ounce each day. Mark, with a baby brother of weight 8lb., found out that the baby had 6 feeds a day of 5 oz. "That's nearly one quarter of his weight," Mark wrote, He then kept a record of the food he ate himself. He found this to be about 1/25 of his weight. "Why do babies and mice, both small creatures, need more food?" the children asked. After much discussion with the teacher they decided that skin area and loss of heat might be part of the reason. The teacher (who had been doing some reading on the subject) suggested that the children should use a mathematical shape to find the relationship between skin area and weight (or volume). They chose inch cubes and used these to build a set of cubes of edges 1, 2, 3 etc. inches. They made a table showing the skin area/volume relationship. Here is the beginning:-

Edge (inches)	*Skin area (A sq. in.)*	*Volume (V cu.in.)*	$\frac{A}{V}$
1	*6*	*1*	*6*
2	*24*	*8*	*3*

This showed that the skin area/volume rate was halved when the edge was doubled, and a graph helped to clarify this still further. Some of the children found this idea difficult and repeated the experience using identical cuboids instead of unit cubes. Using the cubes the children discovered several other sequences and patterns concerning the perimeter and area of one face. From the number patterns they predicted the type of graph (straight line or curve). They found, too, that there were several other applications of these relationships in biology. (Geography provides many other important applications: scale maps and the globe, surveying, etc.)

(f) Limits

The last topic I want to consider is limits. I have already given many examples of this in the constant-perimeter and area section, and in symmetry. The children who continued their investigation to 450 million sides had first-hand experience of a mathematical limit when they realised that, even with that polygon, they would never reach 180° for the angle. But the concept of a limit can be experienced and understood by younger children.

I asked a group of 8 year olds in Ontario to make the largest square they could from a sheet of paper. After some false starts this was done and the extra piece removed. But in doing this the children had folded the square along one *diagonal and obtained a triangle. "What shape is it?", I asked. "A right angled triangle" they replied. "How do you know it is a right angle?" "It's the corner of a square", was the answer (showing that the children, too, understood the difference between intuition and mathematical proof). "Its two sides are equal", said Cathy. "How do you know?", I asked. "Because they match", "Because they are sides of a square", were the two answers given. In matching the equal sides the children had made another isosceles right angled triangle. "It's the same shape as the other". "It's half the other triangle and one quarter of the square" said another. With mounting excitement they continued to fold saying "$\frac{1}{2}$, $\frac{1}{4}$, $\frac{1}{8}$, $\frac{1}{16}$, $\frac{1}{32}$ (words failed them after this). "What will happen eventually?", I asked. "You will get a very wee triangle but we shan't ever get to the centre", said Cathy. Scott remarked that as the triangle got smaller the paper got thicker until it was too thick to fold. At this stage I asked them to unfold their paper. Many comments were made on the pattern of triangles and squares. I asked them to cut the square in two along the diagonal and make the next larger triangle in the sequence. (This was very quickly done.) "Now join with a friend and make the next in the sequence, using four triangles". Before I could do this myself Janet shouted, "Me and Scott will do it and we can go on and on till we fill the room".*

This example exhibits several characteristics I want to emphasise. First, the children had an idea of a limiting sequence of numbers as well as of shapes. Also, the concepts of symmetry and similarity were introduced. But the material itself was simple. The work was not planned and I was carried on by the rising interest of the children.

In-Service Training

21. I have already referred to the need for teachers to know more mathematics, but important though this is, it is not enough. If teachers are to provide their children with opportunities for discovering mathematical ideas they must be convinced that this is possible from their own first hand experience. We have, therefore, developed our in-service training to provide teachers with opportunities for discovering by investigation the mathematics they need to know. They work in groups of eight, each with a tutor (often a leader, teacher or lecturer from a teachers' college). Each group contains infant, junior and secondary teachers. This contact is very helpful. There are various patterns of in-service training, for example, an initial 3 or 5 day course followed up at teachers' centres one session a week (afternoon or evening) for 8 to 16 sessions. It is important that most of the teachers' time is spent in working either at their own level, or in preparing work for the classroom. Often each group will cover the age range 5 to 16.

22. Teachers' centres were first set up in Britain early in 1963. These were provided by Education Authorities to give teachers opportunities of meeting in small groups at regular intervals to study mathematics and other subjects. Nearly 100 centres were set up in areas associated with the Nuffield Mathematics Teaching Project or the Junior Science Project, when these began in 1964/5. Today there are more than 300 centres; many of these now provide for the study of other subjects as well. Some of the centres are used as classrooms by day and as centres in the evening; others have been specially equipped.

23. Teachers from infant, junior and secondary schools frequently meet at these centres to study various topics in mathematics, to prepare work for the classroom and later to bring children's work to compare and discuss. In this way they are able to cover different aspects of mathematics and to learn from each other. Frequently the leaders are teachers although lecturers from colleges of education are usually called in to help.

Modern Mathematics

24. I have not mentioned so called "modern mathematics". The new ideas emphasise both the structure and the unity of mathematics and have helped us, too, to take a new look at more traditional topics in arithmetic, geometry and algebra. A knowledge of the foundations of number is of importance to all teachers. Many of the new ideas are already included in the curriculum because these are natural activities of young children, e.g., matching one-to-one, sorting (into sub sets), ordering relations. Communication of results has become surprisingly varied; language, mapping (a very useful way of recording), tabular forms, representations using three-dimensional material or pictures, and graphs of many kinds. These topics do not have to be specially introduced and given a new vocabulary and symbolism (except perhaps to use the word 'set' instead of group or collection). The formalisation of the new ideas is surely an activity which should occur at the secondary stage for most pupils but the concepts themselves, and a knowledge of them by teachers, are highly desirable if sound foundations are to be laid. In brief, a knowledge of modern mathematics is very useful for us but our children will come to no harm if they do not meet its more formal aspects until the secondary stage. It is essential that teachers should not be encouraged to experiment with new material *unless* they have sufficient knowledge and confidence.

25. In Britain the Nuffield Mathematics Teaching Project is experimenting with topics from modern mathematics in schools in more than a hundred areas. At their local centres teachers are asked to comment on their experiences of these topics.

Parents

26. But there are others concerned in new ways of learning, for example, parents and lecturers at colleges of education. The best way to initiate parents is to let them learn as we are encouraging their children to learn – actively, through real experience and by means of challenging questions. Many teachers' centres have fulfilled their function as parents' centres also.

Colleges of Education

27. It is fortunate that in Britain Her Majesty's Inspectors and lecturers from colleges have had annual joint conference on mathematics since 1956. We have worked together, too, on a number of teachers' courses. Many lecturers feel, as I do, that students, like fully fledged teachers, require to learn mathematics in the same active way as the children they teach. This is time-consuming and some principals are not yet aware of the time required to do this and to help to remedy deficiencies in the students' mathematical background. We still have a long way to go.

Conclusion

28. I want to end with my favourite story of children.

> *It concerns Peter, a ten year old, whose IQ was said to be 110 and who would not normally have been transferred to one of our grammar schools. Until he was ten Peter and some 40 others in his class had had a traditional education in arithmetic. In his last year in the junior school he was taught mathematics by a pioneer teacher. At first the children were hesitant and insecure. One day, however, Peter asked: "If I run a truck down the ramp and time it, then jack the ramp up twice as high, would the time be halved?" The teacher suggested that Peter and two friends should investigate the problem which occupied them for a whole morning. This was the first time Peter had taken the initiative. Some time later Peter brought the teacher his graph of how squares grow ($y = x^2$) and said he wanted to find two patterns. One was the pattern of areas under the curve at 1 unit intervals. The teacher was very surprised and asked, "But why do you think there is a pattern, Peter?" He replied,* **"In mathematics there's always a pattern, you've only got to look for it".** *This started a six months' piece of work during which Peter, like Archimedes before him, discovered the calculus (both integral and differential). Peter had only one method at his disposal: he drew the curve ($y = x^2$) carefully, counted squares and approximated to find successive areas. At one stage he was faced by the sequence:*
>
> $$\frac{1}{3}, 2\frac{2}{3}, 9, 21\frac{1}{3}.$$
>
> *This held him up until he decided to multiply by 3 – when he saw the pattern immediately. Peter's excited cry: "It's x cubed over 3", was reminiscent of Archimedes' "Eureka".*
>
> *Peter's parents were totally unable to help him at home, where much of his work was done. Indeed, Peter developed too much independence to have allowed this. When his work was complete I talked to Peter about it. He had studied the sequence $y = x^2$, $y = x^3$, $y = x^4$. I asked him if his findings would apply if he continued the sequence in the other direction. He replied, "$y = x$, $y = 1$", then said, with mounting excitement, "I am going away to try this myself. I am afraid you will tell me", and he rushed out of the room until he had completed the problem. When he returned he said, "Smashing morning! It works that way too".*

29. This story is significant for many reasons. Peter was not the most intelligent in his class, but once his imagination was fired he had the creativity and persistence to complete an outstanding piece of abstract mathematics. The problem was his own, therefore Peter was intensely interested in solving it, but he would not have asked the question had he not learnt mathematics in a creative, and permissive way. His first hand experience had made him aware of the order and pattern in mathematics. He possessed the skills he required to complete the solution. So Peter's work fulfils my three aims: releasing children to think, letting them discover the patterns in mathematics, giving them the skills. This way of learning mathematics is causing a revolution

in Britain, at all levels and with all abilities. We have a long way to go, but we shall not fail if we can convince all our teachers and equip them to work in this way.

REPORT OF WORKING GROUP A.1.

Chairman: Mrs. E.M. Williams (Britain)

Introduction

30. Group A.1 faced a double task in considering both method and content. To achieve both aims the group concentrated on the change of approach without specific regard to content and the introduction of new material, which would be taught by the newer methods. There was a ready acceptance of the change of approach from a predominance of class instruction to a proportion of active investigation by the children themselves. These changes are in line with modern thinking in education generally and their successful application in mathematics may lead to similar changes in other aspects of the curriculum.

31. There are major difficulties to be faced in adopting more modern methods. First there is a shortage of teachers adequately qualified for the new mathematics programmes; secondly there is a shortage of specially designed materials. Thirdly, where selection for secondary education depends on an examination the mathematics syllabus in primary schools may be restricted to the requirements of the examination. This report therefore presents a compromise between the desirable goal for the future and what is practicable at the present stage in different countries.

32. In defining the range of the enquiry the group analysed the limits of primary education in the areas represented. It was found that the majority primary education extended from 5/6 to 11/12 years of age. Some countries still have schools covering the age range 5 to 15. For the purposes of this report only the first six years of primary education will be considered.

Active Experience as the Basis for Learning Mathematics

33. In active learning the emphasis is shifted from instruction by teachers to investigation by children themselves. The teacher creates opportunities for children to invent and discover by providing materials for them to handle which he or she knows from personal experience will cause them to ask questions and to initiate their own explorations. But it is not always necessary to give children real materials; sometimes they will be stirred to ask about things they have observed and to pursue enquiries without prompting from the teacher. At other times their investigation of problems associated with shapes or numbers will lead them to invent a new problem. In brief, methods of active learning have caused a shift from over-emphasis on computation to a discovery of patterns and relations which will be found in experiences of space and number, in measurement of many kinds, in mechanical devices and in natural forms.

Arguments advanced in support of active learning

34. (a) Classroom experience in many parts of the world has indicated that children learning by means of their own explorations learn more

thoroughly. "Every teacher knows that a young child will learn more readily when he is engrossed in an activity". (Introduction to a West Indian Guide for teachers).

(b) Children of all ages and of all abilities from the least to the most able profit from these methods. Children who are slow at calculations may readily see spatial relations which will, in turn, help them to recognise number patterns.

(c) Children enjoy mathematics brought to them in this way and work with enthusiasm. The feeling of certainty that what they have discovered is right gives them confidence in their own powers.

(d) When these methods are applied to arithmetic and children are encouraged to devise their own methods for written calculations, they understand what they are doing and far less time is required for practice in isolation from experience.

(e) Since children are learning through questions and investigations the emphasis is on problem situations. The transition from the use of real materials to the consideration of verbal problems is an easy one because children have formed vivid mental images both of patterns they have encountered and of the ways in which they found or made them.

The teacher's role

35. Children need to handle real material; sensory experience is valuable in arousing their curiosity. The teacher selects material which has mathematical potential, observes the child's use of this and asks questions, when necessary, to help the child's thinking. Sometimes children start their own enquiries, at other times the teacher will need to ask an open-ended question or suggest an experiment to start them thinking. The form of the question asked is important. Open-ended questions stimulate a variety of suggestions. Directed questions rarely allow opportunity for inventiveness. (See The Teaching of Mathematics at Primary Level, Lead Paper by Miss E.E. Biggs, paragraph 10, for examples of directed and open-ended questions).

The value of discussion

36. When children are deeply involved in an investigation they will want to communicate their findings to their teacher and to their companions. Although there are various means of communication such as drawing, making models and writing, it is important that children should have opportunities to talk to other children as well as to their teachers either about their successes or about their perplexities. Such discussion may reveal language difficulties. This may be due to lack of opportunities for conversation in the home; or it may be that the medium of instruction is not the vernacular. If the mother tongue does not have the relevant mathematical phrase it may be necessary to provide the appropriate phrase in the ultimate language of instruction. For all children first hand experience and discussion will extend their vocabulary, enlarging the number of words they understand and words they can use spontaneously.

37. In some countries there is a tradition that children are not expected to speak unless they are spoken to; talking may even be regarded as a form of rudeness. In such circumstances free discussion will only occur if conversation is encouraged in the classroom. Since by discussion children clarify their thoughts and increase their command of language, such conversation is essential. This implies a freer atmosphere in the classroom than is found in formal class teaching.

The necessary materials

38. The materials required for an active programme are selected by teachers with the children's mathematical needs in mind. Through their choice the teachers are providing guide lines for their children's development. Some of the things needed will be found in the immediate environment and both children and teacher may be concerned in collecting these. But not all material can be provided in this way and it was strongly urged that Authorities should make money available for certain basic materials and equipment.

39. It is not likely that funds will be sufficient for all requirements. It was therefore suggested that teachers should be encouraged to make or improvise as many of the things they need as possible. If accommodation and facilities for this purpose are provided teachers will be able to share, possibly at a special centre, ideas about the use of local materials. Various ingenious schemes have been devised in several countries in Africa and elsewhere to produce equipment from local materials on a large scale.

40. Of course books are important. There are many books on the market for teachers and for children, including reference books of information. These can be a useful source of ideas for teachers as well as children. Each school should aim at making a varied collection using all available funds. Many teachers will require textbooks to help them to provide the varied activities which children will need and also for practice in computation.

A suggested list of useful materials

41. *(a)* Shapes

Boxes, round tins and other containers; leaves, shells, fruits, seeds, flowers, inch and centimetre cubes, beads, balls, globe, mirrors.

(b) Sorting, matching and counting materials, structural materials

Seeds, stones, shells, small plastic toys, cubes, rods (length 1 to 10 units), number track, a number line painted on the wall or marked on the ground in a variety of relevant units, beads on a taut horizontal string.

(c) Measurement

Length : *Canes or softwood, string or fibre, homemade trundle wheel (yard or metre in circumference).*

Weight : *Balance scales, springs coiled wire (hair rollers, extension springs tendrils of climbing plants).*

Lever: *straight stick or rod to suspend, or to balance on a small wedge of wood, metal washers.*

Capacity: *Pots, gourds, cocoanut shells, pails, tins.*

Time: *String and bob for pendulum.*

Area : *Various materials to cover a plane (or curved) surface or a model, e.g. textiles, newsprint, leaves, seeds, squares, identical triangles, etc., mats, geo-boards or peg boards made of softwood with pins or pegs.*

Rotation : *compass, geared wheels, protractor, homemade clinometer.*

(d) Constructional Materials
Commercially produced strips and bolts or drinking straws, building blocks of various shapes and sizes (off-cuts), homemade level.

(e) Materials for Communication and Recording
Squared paper (in inch and centimetre uints), coloured paper and card, newsprint or other cheap paper, paints or dyes, brushes, coloured pencils or felt-tipped pens, abacus and rings of dough or paper beads, scissors.

Introducing Active Learning Situations in the Classroom

42. (1) The headteacher is a key person in the introduction of new ideas and could give a lead in his school. Exchange of ideas among the staff in the school is of first importance. To gain confidence teachers need to start with a small group of children. It would be ideal if the headteacher took the remainder of the class; alternatively these children could be set more formal work while the teacher was working with the selected group. The teacher has to convince himself that children using materials can discover mathematical patterns and relations for themselves without being given precise instruction. One method of organisation is to initiate a different group each day until teacher and children feel confident in this way of learning. The number of children included each day can then be extended.

(2) Teachers should be fully aware of the reasons for introducing new methods before they try these. They need to experience the active learning of mathematics themselves before they attempt this approach in their own classes. After teachers have attended an initial course which provides such opportunities, they need continued help while still experimenting in their classrooms. This help could come from leader-teachers, lecturers from teachers' colleges or advisory teachers. In some of the countries represented at the Conference full-time advisory teachers have been appointed to help in primary schools.

(3) Teachers embarking on a new syllabus find help in guides issued by education authorities and others. In some countries the syllabus and guides have been drawn up by groups of teachers, tried out in their classrooms, subsequently amended, and then approved for use. When introducing new methods it would be a great help if teachers could visit schools where good teachers were already using these methods.

(4) All those concerned with education should be made aware of the new methods: supervisors, headteachers, lecturers in teachers' colleges and parents. The

best way to help them is to organise meetings in which they are able to try for themselves, to learn new ideas by the new ways.

(5) When learning from experiences children cannot be kept in rigid rows. If classrooms are over-crowded children can work out-of-doors when conditions are suitable. For the future, movable flat-topped desks should be provided for older as well as younger grades. Extra tables, shelves, and a cupboard (steel where necessary) would be needed for display and storage. Additional display space is required for children's finished work. In planning new school buildings, Authorities should bear in mind that a classroom programme which involves considerable activity for the children requires different kinds of buildings and equipment.

Fundamental Content

43. Children's first experiences of mathematics should include opportunities for imaginative construction with objects which attract them. Through such activities they grow aware of the characteristics which things possess, such as colour, shape, heaviness and texture, and of likeness and difference in the things they are handling. Relevant vocabulary should develop easily. The range of activities offered at the early stages provides the basis of early ideas of number and spatial relations and the measurement of continuous quantities. They will also help a child to form mental pictures of his actions which will later make mental operations possible. The various aspects of mathematics should develop side by side although, for convenience, they are considered separately in subsequent paragraphs.

Foundations of Number

(1) The pre-counting stage

44. At the pre-counting stage many opportunities for the necessary sorting and matching experiences will arise naturally at home and can easily be contrived at school; for example: sorting bowls, pebbles or packages for colour, shape, size etc. Children usually notice that one of the sets has more things in it than another and the idea of comparing arises. Matching will also have arisen naturally at home and in school e.g. in setting out bowls for a family meal and in arranging mats or chairs for each child. Matching (one-to-one correspondence) can therefore be used to compare the sets. During these experiences the language which expresses the relation of inequality will normally arise before that of equality and will be used more frequently e.g. *more and fewer* before *as many as* (and later *more and less* before *as much as* when dealing with quantities). Other ideas of correspondence arise when talking about pairs of sandals and the people they belong to, and the children belonging to a family (many to one); a cat and her kittens or a hen and her chicks (one to many).

(2) Beginnings of counting

45. At this stage children will recognise the smaller numbers, perhaps 1 to 5, and will use their names, matching these number names to sets of real objects. Before children can really count objects they must have experience of order; for example: heavier and lighter, longer and shorter, arranging three children in order of height.

46. The idea of cardinal number can be extended beyond the easily recognised numbers when children can make a sequence of sets (beginning with one object),

each of them with one more member than the previous set. For example, a class of children arranged themselves in order according to the number of children in their families. Each child drew all the children in his family, naming each child and including himself. The families were arranged in sequence with equivalent sets (families with the same number of children) placed together. Children who can recognise equivalent sets must already have realised that the number of objects in a set does not depend on their arrangement.

47. Experiences of order also occur when children are comparing the contents (in spoonsful or cupsful) of a set of utensils, the weight of a set of stones (shown on a stretched spring or by using identical washers on balance scales), the lengths of their own feet (marked on a strip of paper), so that they form a sequence. After this children are soon able to count in spoonsful, washers or unit strips. Further experience should include keeping a record of the weight of a growing animal and the height of a growing plant. This, too, can be done by direct transfer of the daily height to a vertical strip.

(3) Sequences

48. The naming of positions in a sequence: first, second, third, etc., arises in dealing with quantities as well as with sets, e.g. order of entering or leaving the classroom, the order of children scoring points in a game.

49. In addition to arranging a set of objects in sequence, children should be asked to make a variety of patterns with them. They will then discover the patterns which are characteristic of any particular number, e.g. the rectangular pattern made by the even numbers or the pattern made by packing seven discs close together.

A sequence of sets with cardinal numbers 1,2,3,4, etc. is well illustrated by a child stepping along a line and marking his steps as he goes. Such movement can be shown on a graduated number line which will then be available to extend the child's counting.

(4) Operations

50. The handling of sets of objects, of continuous quantities (length, weight, capacity, time) and of money, also gives rise to operations with numbers. These will include matching and comparing (by subtraction and division); combining and separating will involve all four basic operations. These operations may arise in any order. For example, finding one-half (or one-quarter) of a yard of fibre illustrates the sharing or fractional aspects of division. Finding how many 9″ lengths can be cut from a yard of fibre involves the measuring or subtraction aspect of division. Both these experiences can be reversed by the teacher by asking the question: "What length of fibre do you need altogether to give a 9″ length to each of 4 children?"

51. At first, recording will be in pictures or in the child's own words orally or in writing. Later the teacher can introduce the shorthand mathematical way of recording, e.g. $9 + 5 = 14$ to match the experience of 5 more children joining a set of 9.

52. A number track and rods (1 to 10 units long) offer further opportunities for experience with the operations which will now be more closely concerned with the numbers themselves. Movement along the number line illustrates the operations of addition and subtraction, multiplication and division and their inter-relationships.

(5) Tabulation

53. The tabulation of results found from a number line (and from other experiences) gives an ordered mental picture which helps children to recall the pattern of a sequence and makes a contribution to the memorisation of number facts. A confident knowledge of such facts (addition and subtraction before multiplication and division) is essential before children can undertake the calculations which they need to write. By careful planning of experiences and questions a teacher can allow children to devise their own methods for written calculations, for example, in long multiplication and division.

54. The relations shown in the tabulation of a sequence, for example the set of multiples 0,3,6,9, etc., can be represented:

(1) by arranging unit cubes and rods;
(2) by a block graph;
(3) by a column graph.

Finally, the relationship can be extended and can be represented either by joining any pair of corresponding points on two parallel number lines or by a *continuous line* referred to two number lines at right angles.

(6) Laws of Arithmetic

55. Awareness of these laws may grow from examining tabulations. For instance, addition or multiplication facts can be organised as a square array or table (Fig.1) from which symmetry about the diagonal will show the commutative law of addition:

2 + 4 = 4 + 2 = (6)

3 + 5 = 5 + 3 = [8]

+	1	2	3	4	5	9
1	2	3	4	5	6	....
2	3	4	5	(6)	7	....
3	4	5	6	7	[8]	....
4	5	(6)	7	8	9	....
5	6	7	[8]	9	10	....
.	.	.	.	.	.	....
.	.	.	.	.	.	....
.	.	.	.	.	.	....
9	.	.	.	.	.	....

Fig. 1

56. The corresponding multiplication square (Fig. 2) will show the same law for the operation of multiplication. This brings out the rectangular pattern characteristic of a multiple.

2 × 3 = 3 × 2 = [6]

6 × 1 = 1 × 6 = [6]

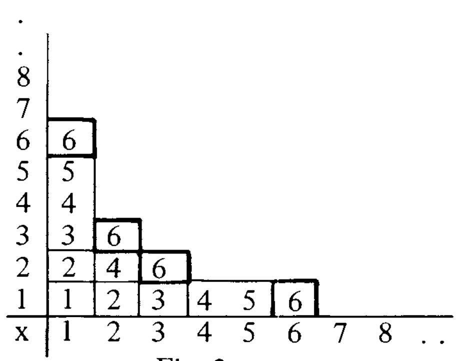


Fig. 2

(7) Systems of notation

57. Experience with other number bases helps children to understand the structure underlying the denary system as well. The

first encounter with such bases may be through packing a set of objects into square trays e.g. 5 by 5, rows of 5 and any odd ones. The use of unit cubes, rods and layers (as in multibase blocks) gives children one type of manipulative experience which leads into the more abstract representation on an abacus. The binary scale with its two digits, 0 and 1, has a special fascination for children (and adults); the fact that it is associated with the action of a computer gives a further attraction. Experience with this notation gives children a delighted appreciation of some of the number patterns which can be discovered.

58. The long sequence of powers of a small base such as four: 4, 4×4, $4 \times 4 \times 4$, $4 \times 4 \times 4 \times 4$,... may lead children to invent a shorthand notation related to the number of factors in each power. The conventional exponential form 4^1, 4^2, 4^3, 4^4,... will then be used.

Extensions to the use of numbers

(1) Fractions, decimals, ratio

59. The more experience children have of measuring, the more likely they are to be familiar with fractional parts of a unit. Measurement of the quantities mentioned earlier provide instances of "the bit over". The division of a quantity by a number such as 4 (finding a quarter of) leads rapidly to the idea of a fraction written as $\frac{1}{4}$. A wide variety of experiences of paper folding and cutting, of sharing a quantity or partitioning a set, is needed to establish the equivalence of fractions, a relation which is required for any calculations necessary. Children can devise their own methods for these calculations. The methods children invent are usually simpler than those we teach them.

60. Because we have to use two numbers to express a fraction other approaches have been tried out. Some teachers use the notation of fractions to mean the two operations multiplication and division; e.g. $\frac{3}{4}$ means divide by 4 and multiply by 3. Alternatively, if the two numbers are shown as co-ordinates on a graph we see the fraction represented by the ratio of the two numbers 3 and 4 or as the ratio of the co-ordinates (4,3). The ordered pair (4,3) is then used to represent the fraction $\frac{3}{4}$. (3,4) is sometimes written to represent the fraction $\frac{3}{4}$.

61. The equivalent value of 10 cents and a dime is a good introduction to both the idea of $\frac{1}{10}$ and to the decimal notation. 100 cents to a dollar extends the notation. The use of metric measures carries the notation further and it is hoped that countries which do not use the metric system will make plans for its adoption. The calculation and interpretation of percentages is easy when decimals are used. Integral percentages are hundredths and can be read immediately from the second decimal place. A representation on squared paper gives a useful ready reckoner for finding a percentage (e.g. 37/45) on two axes at right angles, provided the idea of ratio is understood.

62. One of the most valuable uses of fractions is to express a ratio, for example, the representative fraction used to show the scale of a map. Children from about 6 years of age will spontaneously use a rough scale in the models and drawings they make. When building two corresponding models using centimetre cubes for one and

inch cubes for the other, children will expect corresponding lengths in the two models to be in the same ratio and will verify this. (The volumes and areas of the two models will *not* be in the ratio of their lengths!). When accurate drawings are made the ratio of corresponding lengths can be expressed in fractional form. This gives rise to the word *rational* used to describe numbers expressed in this way. When rational numbers are represented on a number line children realise that any natural number can be divided by another number.

(2) Positive and negative numbers

63. When children try to see whether subtraction is commutative they sometimes think of extending the number line to the left of zero and so invent the negative numbers. The idea of positive and negative numbers may arise earlier through such practical experiences as reading temperatures above and below zero, finding the height above and the depth below water level, walking up and down stairs from the first floor or noticing the corresponding distances of reflected points from the axis of reflection.

(3) Statistics and Averages

64. Many enquiries initiated by children come from situations in and around the school, e.g. the number of children wearing certain types of clothing (blue shirts, etc.), the number of vehicles passing the school in a given interval of time, the size of shoe or length of foot. Usually the enquiries begin with a small group of children, then extend to the class and later may involve several classes and make the use of large numbers inevitable.

65. A block graph in which the children themselves are the units can be made, for example, when children choosing coloured rods from a container arrange themselves in rows according to the colour chosen. Representation of information collected in these enquiries should include:-

(a) block graph (in which the spaces are labelled). The number (frequency) of children in each set is marked on the vertical axis and heights of blocks can be compared.

(b) column graph (in which vertical lines replace the blocks, the points on the horizontal axis must be labelled).

(c) points marked at the tops of the columns. In a statistical enquiry involving a large number of children the points may be joined to show a trend. A clear distinction should be made between such a distribution graph and a continuous growth or relationship.

66. From these representations children usually comment on the most popular colour (called the *mode*) and on the range of the distribution. In some enquiries, for example into shoe sizes, the mode has relevance whereas the arithmetic mean (usually called the average) does not.

67. Where the arithmetic mean is relevant children can be asked to make a careful guess at this and to check their guess by calculating the deviations above and

below. Children sometimes use + and – spontaneously to denote quantities above and below both the estimated and the calculated mean.

Space and Shape

68. The study of movement, change and development is basic to the child's understanding of the real world around him. The transformation of the set of numbers (1,2,3,4,) into the set (3,4,5,6) by the operation of adding 2 is matched by the transformation of the points representing the numbers by a movement of 2 along a number line. The idea of transformation also gives an opening for imaginative pattern-making when a unit shape is translated along a line to make a border pattern.

69. Children will become familiar with certain basic two-dimensional shapes (e.g. squares, rectangles, circles, triangles) by handling a variety of three-dimensional objects which they should be encouraged to collect. By handling and comparing various shapes, children discover their distinctive properties, for example the likenesses of and differences between cubes and cuboids; squares, rectangles and rhombuses (diamonds); cylinders, spheres and cones.

70. The following activities give varied opportunities for investigation.

(1) Constructional activities

(a) building structures with unit cubes and unit cuboids,

(b) making patterns with unit squares, unit rectangles, unit triangles etc.

(c) making frameworks with split bamboo canes, drinking straws or meccano,

(d) model-making in card and other materials: sheds and buildings as well as the regular solids (tetrahedron, cube, etc.).

Interesting shapes such as bandstands, drums, balls, ice cream cones, the scoop for serving ice cream, pyramids could be made in the classroom and their properties investigated.

(e) making or investigating toys, for example kites, tops and hoops, doll-dressing. Mechanical toys and devices are not easily available in all countries, but some could be home-made, for example model aeroplanes, wheeled toys and an inclined plane (for experiment), go-carts, slings.

(f) Games such as cricket, football, basket ball and tennis can make children aware of the path of a moving object.

(g) A geo-board is useful for discovering relations of areas of different shapes.

(h) Paper folding, experiments with paint and mirrors lead to investigations of symmetry and enlargement (similarity). Children sometimes invent "co-ordinates" for themselves when they describe the position of buried treasure on an imaginary island they have drawn. When trying to find the treasure they realise the value of order in the pairs of distances used to fix the position of the treasure and ordered number pairs are then accepted. Co-ordinates can also be used in symmetrical reflections and other transformations.

(2) Mathematical ideas derived from these activities.

(a) The properties of various two-dimensional shapes, from making and handling frameworks with 3, 4, 5 etc. sides, using first equal then unequal strips; the rigidity of the triangle. Sequences of numbers from a series of frameworks of regular polygons made rigid by longer struts.

(b) Ideas of angles from a child's own observations of the hands of a clock and of other changes of direction. Angle properties of parallel lines and angle sum of a triangle from patterns made in paper folding. Rotation and gear ratio from cog wheels of different sizes.

(c) Mathematical similarity (enlargement of scale) in three dimensions, by building a sequence of cubes from unit cubes. Paths on the globe and on the earth. Similarity in two dimensions by building a sequence of squares from unit squares (and other shapes). Maps. Recognition that some shapes (e.g. cubes, spheres, squares) are always similar.

(d) Volume of cubes, spheres, cylinders, cones and pyramids from practical experience, for example, comparison of the weights of clay used to make the objects, or of the water contained by or displaced by them.

(e) Area: approach through irregular shapes, for example comparing the area of two leaves; counting the number of yam heaps covering each of two fields. Many activities lead to the understanding and application of area, such as gardening, weaving mats, doll dressing, making pictures with scraps of fabric. The making of nets of solids gives a useful link between two and three dimensions.

(f) The speeds of various moving objects compared with the child's own speeds.

(g) Transformation: The making of decorative patterns leads to an awareness of the various movements of a unit shape required to make different types of pattern: translation along a line, rotation about a point, reflection. Correspondences of points, and relations between lines and angles, can be investigated. The basic idea of congruence can be firmly established by means of these experiences.

Graphical representation

71. Representation on two number lines (axes) at right angles has already been mentioned. Certain types of relations occur frequently so that children come to associate the shape of a continuous graph with the corresponding number pattern. These include:

(1) Straight lines

(i) The pattern of the graphs of multiplication tables of 2, 3, 4, etc. as straight lines of increasing steepness. (Sometimes children ask if tables must stop at zero and so invent the negative numbers and extend the axes).

(ii) The perimeter/side graph for squares.

(iii) The circumference/diameter graph for circles.

(iv) A graph showing constant speed.

(v) The extension of a spring as weights are added to one end.

Children sometimes predict that the graph of the multiplication tables or perimeters of squares will be straight line because of the equal differences.

Edge of Squares	1		2		3		4		5
Perimeter	4		8		12		16		20
Differences		4		4		4		4	

(2) Graphs of squares and cubes

(i) The area/edge graph for the squares.

(ii) The area/diameter (or radius) relation for circles found approximately by counting squares. The number pattern of the area of squares is soon recognised.

Edge of Squares	0		1		2		3		4		5	-	-
Area of Squares	0		1		4		9		16		25	-	-
Differences		1		3		5		7		9			

Some children realise that because these differences are not equal, the graph will not be a straight line. Because the differences increase and form the odd number pattern they expect a rising curve.

(iii) The volume of cubes and of spheres will be investigated and will lead in a similar way to a curve.

(3) Constant product curve

(i) Examples of the constant product pairs have already arisen, e.g. in the multiplication square.

(ii) Dimensions of rectangles made with 24 squares. The complete set of rectangles with integral dimensions can be cut out and put in order: 1 by 24, 2 by 12, 3 by 8, 24 by 1. When children are asked to arrange these in order taking up the least possible space, they sometimes overlap them and then recognise the similarity to the pattern in the multiplication square.

Ordered pairs, (1,24), (2,12), ... , (24,1), with the relation $wl = 24$ give a continuous curve.

Links with other Subjects

72. Several very interesting projects were described by various delegates. Such projects gave children more confidence in their own powers particularly when the work was displayed. These topics normally arise in other aspects of the curriculum, e.g. social studies, science, art and crafts. Teachers with a sufficient knowledge of

mathematics can fully exploit the mathematical possibilities in activities of this kind. These should be carefully followed up and developed. The following examples were quoted;

(i) Social Studies: The banana industry had been studied in different ways according to the country of origin. Model making (sheds, trucks, boats). Different packing methods led to different developments in the classroom.

(ii) Art and Crafts: For example, construction of bird cages with palm pith; patterns designed for decorative purposes in craft of various kinds.

PLENARY DISCUSSION OF REPORT OF WORKING GROUP A.1

Survey and Comments

73. The Chairman introduced the report with explanation of its considerable length. The early years were by far the most important, partly because many more children were in primary schools than in secondary schools and partly because the only education some thousands of children would receive would be a primary one. For this reason primary education was not only a preparation for secondary education; it must have its own particular goals. It was at the primary stage, also, that the greatest impact of the present changes would be felt. An emphasis on the development of children's thinking might have an appreciable influence on man's future power of decision-making.

74. The first part of the report deals with the changing approach. Then the content for the early years is discussed in some detail. Finally there is an outline of topics suitable for the later stages which will extend the range of mathematical thinking.

75. Subsequent discussion on the report centred on

(1) the problems arising when the mother tongue did not contain words appropriate to certain mathematical experiences

(2) the influence on the primary stage exerted by selection examinations for secondary education. These examinations aggravated the difficulties of adopting a fundamentally different mathematical programme since too much time was often devoted to preparation for such examinations.

76. Delegates expressed some anxiety lest the emphasis on children's practical experiences should lead some teachers to treat practical activities as ends in themselves. It was pointed out that the early sections of the report stressed the thinking that can emerge from children's individual experience and the need for teachers to guide their pupils – by questions, discussion and various forms of representation – to a clear awareness of the mathematical patterns that have been disclosed by their actions.

77. It was agreed at this plenary session that the editors should do some re-editing of the paper in the light of the discussion, and some modifications were made.